DEPARTMENT OF ELECTRICAL ENGINEERING

COLLEGE OF ENGINEERING

UNIVERSITY OF WASHINGTON

A STUDY OF SOME NEARLY OPTIMUM SERVOMECHANISMS

By

T. M. STOUT

TECHNICAL REPORT

PREPARED UNDER CONTRACT Nonr-477(02)

OPTIMUM DESIGN OF DELIBERATELY NONLINEAR SERVOMECHANISMS

(NR 374 371)

For

OFFICE OF NAVAL RESEARCH

EE DEPARTMENT REPORT NO. 10

November 28, 1952

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Technical Report
Prepared under Contract Nonr-477(02)
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Office of Naval Research

EE Department Report No. 10
November [illegible]

Department of Electrical Engineering

University of Washington

ACKNOWLEDGMENT

The work described in this report was made possible by the support of the Office of Naval Research under Contract Nonr-477(02), Project NR 374 371, Optimum Design of Deliberately Nonlinear Servomechanisms.

C. N. Shah assisted with the preparation of the figures.

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ABSTRACT

A theoretical study has been made of the behavior of a recently-proposed relay servomechanism in which the sense of the torque applied to the output shaft is determined by a comparison of the error and the square of the error rate. Response time, integrated absolute error, and integrated squared error for a step input are used to measure the degradation of the response as a result of viscous or Coulomb friction, or imperfect design of the switching device. It is concluded that moderate amounts of friction or slight errors in the switching device can be tolerated with only a small sacrifice in the performance.

A STUDY OF SOME NEARLY OPTIMUM SERVOMECHANISMS

Introduction

Relay servomechanisms have been used and studied[1-7] for a number of years. In the usual system, the sense of the torque applied to the output shaft is controlled by a relay whose operation is based on the sign of the error. Such systems can be made small and simple, but have a tendency to oscillate at small amplitudes if the system has moderate amounts of dead space, backlash, friction, or time lag. Linear lead networks, which make the torque dependent on both error and error rate, have been used to combat these detrimental effects.

McDonald[8-10], Hopkin[11], and others[12] have recently proposed that the usual relay system be modified by the substitution of a nonlinear lead network or "anticipator" for the conventional linear lead network. In the proposed system, the sense of the torque is based on a comparison of the error and the square of the error rate. For a step input and neglecting all friction torques, this system uses maximum accelerating torque until the error is reduced to half of its original value and maximum decelerating torque until the error is reduced to zero.

The proposed system is an "optimum" system since it gives the minimum response time possible with the specified inertia and maximum torque. Because the response time varies as the square root of the initial error and the description of the response to a step input does not furnish a precise statement of the response for other inputs, the system must be considered nonlinear.

In linear systems, slight errors in the values of any of the system parameters produce only slight variations in the character of the response. This common-sense observation is based on a theorem in mathematics which states that, under certain conditions, the solution of a differential equation is a continuous and differentiable function of any parameter in the equation[13]. While the equations of linear servomechanisms meet the conditions of the theorem, the equations of off-on systems may not. For this reason it seems desirable to examine some of the assumptions on which previous analyses of the proposed optimum system have been based, in order to determine whether the parameters of this system are critical and whether differences between theory and practice might have serious consequences.

The investigations to be described are concerned with variations of this system due to friction or improper design of the nonlinear anticipator. These systems are called "nearly optimum" systems because they become "optimum" in the absence of friction or defects in the anticipator.

Response Criteria

Several criteria have been used to measure degradation of the response due to imperfections in the system.

The speed of response to an abrupt change in the input signal is an important characteristic of a control system. In linear systems, the response time for a step disturbance is defined somewhat arbitrarily as the time required for the error to become and remain less than a specified fraction, usually a few hundredths, of the initial magnitude[14]. This sort of definition is perfectly satisfactory for linear systems whose behavior is usually given in nondimensional variables, independent of the input magnitude.

This definition is not particularly useful in nonlinear systems where the character of the response may vary widely with changes in the input magnitude and the error may actually become zero in a finite time. In the following investigations, the response time is defined as the time required for the error to become zero (if this time is finite) or "negligible", which is interpreted as too small to be detected on a graph. The variation of response time with input magnitude and system parameters is considered.

Ambiguities in the definition of response time can be avoided by the use of other criteria, such as the integrated absolute error,

$$A_1 = \int_0^\infty |e| \; dt ,$$

or the integrated squared error,

$$A_2 = \int_0^\infty e^2 \; dt .$$

These integrals, which also depend on the input magnitude and the system parameters, are finite even when the error takes an infinite time to become zero. Both integrals have been used in theoretical and experimental studies of servomechanisms, linear and nonlinear[15-17].

I. THE IDEALIZED SYSTEM

To provide a standard to which the nearly optimum systems can be compared, the theory of the proposed optimum system will be reviewed and the various performance criteria applied to it.

System Equations

The load presented to the motor by the output shaft is assumed to be an inertia load, and all friction torques are ignored. The motor delivers a torque which is either constant, $\pm T_m$, or zero. The response of the system to a step input is then determined from the equation

$$J \ddot{e} = \pm T_m \tag{1}$$

where e is the error, dots denote differentiation with respect to time, and the sign of the torque depends on e and $\dot{e}$ in a manner to be described. This equation may be integrated directly to obtain the error rate

$$\dot{e} = K_1 \pm \frac{T_m}{J} t \tag{2}$$

and the error

$$e = K_2 + K_1 t \pm \frac{T_m}{2J} t^2 \tag{3}$$

as functions of time, where K_1 and K_2 are constants which depend on the initial conditions.

The output shaft is assumed to be stationary when the step input is applied. Because of its inertia, the velocity of the output shaft cannot change instanteously, and the initial error is therefore equal to the magnitude of the step input. For the acceleration interval, the initial conditions are therefore $\dot{e}(0) = 0$ and $e(0) = e_o$, and the equations of motion are

$$\dot{e} = -\frac{T_m}{J} t \tag{4}$$

$$e = e_o - \frac{T_m}{2J} t^2 \tag{5}$$

At a time t_1, the error is reduced to half its original value, the torque is reversed by the action of the nonlinear anticipator, and the equations for the subsequent interval are

$$\dot{e} = \frac{T_m}{J} t - 2\sqrt{\frac{T_m}{J}}\sqrt{e_o} \tag{6}$$

$$= -\frac{T_m}{J}(t_2 - t) \tag{7}$$

$$e = 2e_o - 2\sqrt{\frac{T_m}{J}}\sqrt{e_o}\, t + \frac{T_m}{J} t^2 \tag{8}$$

$$= \frac{T_m}{2J}(t_2 - t)^2 \tag{9}$$

In Eqs. (7) and (9), t_2 is the time required for the error to become zero or the response time. Plots illustrating these relations are given in Fig. 1(a). A family of curves for different values of e_o is given in Fig. 1(c); similar curves, based on experimental data, are given by McDonald[10] and Hopkin[11].

Response Criteria

The switching time, t_1, can be found by substituting $e = e_o/2$ in Eq. (5); the result is

$$t_1 = \sqrt{\frac{J}{T_m} e_o} \tag{10}$$

The response time, t_2, can be found from Eq. (8) or, more simply, by noting that

$$t_2 = 2\ t_1 \tag{11}$$

$$= 2\sqrt{\frac{J}{T_m}}\ e_o^{1/2} \tag{12}$$

It can be seen by inspection that the integrated absolute error is

$$A_1 = \frac{1}{2}\ e_o\ t_2$$

$$= \sqrt{\frac{J}{T_m}}\ e_o^{3/2} \tag{13}$$

The integrated squared error, somewhat more difficult to evaluate, turns out to be

$$A_2 = \frac{23}{30}\sqrt{\frac{J}{T_m}}\ e_o^{5/2} \tag{14}$$

All three measures of response are functions of J/T_m, as would be expected, and a power of e_o which is consistent with dimensional requirements. With the torque-to-inertia ratio and initial error given in any consistent set of units, the response time and integrated absolute and squared error can be computed directly from these equations. Knowing the variation of these quantities with initial error, the values for any other initial error are readily determined.

Switching

To simplify future graphical work, it has been assumed that $T_m = 1$ and $J = 1$, with the result that $\dot{e}$ varies directly with t and e varies as $t^2/2$. This assumption is equivalent to a change of variable which can be made definite if the need arises.

With the substitution of these numerical values, the error can be written

$$e = e_o - \frac{\dot{e}^2}{2} \tag{15}$$

for $0 < t < t_1$, and

$$e = \frac{\dot{e}^2}{2} \tag{16}$$

for $t_1 < t < t_2$. These equations are the basis of the phase-plane plots given in Fig. 2. It will be observed that all of the trajectories coincide approaching the origin; Eq. (16) therefore gives the values of e and $\dot{e}$ at the time of torque reversal and divides the phase plane into regions of positive and negative torque. Points above or to the right of this curve correspond to negative torque, and points below or to the left correspond to positive torque. The curve appears only in the second and fourth quadrants, and is symmetrical about the origin.

In general terms, the equation of the torque-reversal curve (in the fourth quadrant) is

$$\dot{e}_s^{\,2} = \frac{2T_m}{J} e_s \tag{17}$$

where "s" denotes a switching point. The function of the nonlinear anticipating network is to compare $\dot{\delta}^2$ with the proper multiple of e, thereby determining the position of the system with respect to the boundary, and supply a signal to the relay which will result in the proper torque. The physical network can take a variety of forms[8,10,11].

II, COULOMB FRICTION

Previous studies have not considered the possible effects of accidental Coulomb or viscous friction, although McDonald[8] suggests the deliberate use of Coulomb friction to assist in the decelerating process. While it would be possible to study the behavior of the system with friction and the original switching procedure, it seems more fruitful to consider changes in the phase-plane boundary which will minimize the effects of friction. As will be pointed out, a slight change in the shape or location of the torque-reversal curve will preserve the desirable features of the original system with little sacrifice in performance.

System Equations

If motion of the output shaft is opposed by a Coulomb friction torque T_f which is independent of velocity, the net accelerating torque is decreased but the decelerating torque is increased. The motion is still described by Eqs. (2) and (3), except that T_m must be replaced by $(T_m - T_f)$ during the interval when the output shaft is accelerating and by $(T_m + T_f)$ during the interval of deceleration. To obtain the desired type of response, switching must be delayed until the error is less than half its original value, as indicated by Fig. 1(b).

Switching

The error and error rate will become zero simultaneously if the switching is carried out in such a way that

$$t_1 \left[\frac{T_m - T_f}{J}\right] = (t_2 - t_1)\left[\frac{T_m + T_f}{J}\right] \tag{18}$$

$$\frac{1}{2}\left[\frac{T_m - T_f}{J}\right] t_1^2 + \frac{1}{2}\left[\frac{T_m + T_f}{J}\right](t_2 - t_1)^2 = e_o \tag{19}$$

Equation (18) states that the velocity is continuous at t_1, and Eq. (19) states that the total motion of the output shaft is e_o. To satisfy these conditions for all initial error magnitudes, it is necessary to make the switching boundary

$$\dot{e}_s^2 = 2 \frac{T_m + T_f}{J} e_s \tag{20}$$

$$= 2 \frac{T_m}{J} (1 + T_f/T_m) e_s \tag{21}$$

in the fourth quadrant. Comparing this equation with the previous boundary, Eq. (17), we observe that they are identical except for the factor $(1 + T_f/T_m)$. This means that the square of the error rate must be compared with a larger multiple of the error; only a slight adjustment of the nonlinear network is required.

Phase-plane plots, showing the new location of the torque-reversal curve, are shown in Figs. 3(a) and 3(b) for T_f/T_m equal to 0.4 and 0.8. It will be seen that the switching curve is closer to the vertical axis, reflecting the delay in switching, and that the maximum velocities are reduced, indicating an increase in response time.

Response Criteria

The switching time and response time may be calculated from Eqs. (18) and (19), giving

$$t_1 = \sqrt{\frac{Je_o}{T_m}} \left[\frac{1 + x}{1 - x}\right]^{1/2} \qquad (22)$$

$$t_2 = \left[\frac{2}{1 + x}\right] t_1 \qquad (23)$$

$$= 2\sqrt{\frac{J\ e_o}{T_m}} \left[\frac{1}{1 - x^2}\right]^{1/2} \qquad (24)$$

where, for brevity, $x = T_f/T_m$. These expressions reduce to the friction-free values when $x = 0$ and become infinite when $x = 1$, as expected. When the friction torque is relatively small, $x \ll 1$, the response time is approximately

$$t_2 \cong 2\sqrt{\frac{J\ e_o}{T_m}}\ (1 + \frac{x^2}{2}) \qquad (25)$$

For $x = 0.2$, the response time is increased about 2 per cent; for $x = 0.8$, the increase is 67 per cent. The same values may be obtained from plots of error as a function of time, like those of Fig. 4.

That a moderate amount of friction is not too serious may also be concluded from consideration of the integrated absolute and squared error, although these integrals increase more rapidly with x than the response time does. Some messy but straightforward integration and algebra establishes that the integrated absolute error is

$$A_1 = \sqrt{\frac{J}{T_m}}\ e_o^{3/2} \left[\frac{1}{1 - x^2}\right]^{1/2} (1 + \frac{x}{3}) \qquad (26)$$

and the integrated squared error is

$$A_2 = \frac{23}{30}\sqrt{\frac{J}{T_m}}\; e_o^{5/2} \left[\frac{1}{1 - x^2}\right]^{1/2} \left(1 + \frac{10}{23}x - \frac{1}{23}x^2\right) \qquad (27)$$

These equations also reduce to the correct values for $x = 0$, become infinite for $x = 1$, and contain the factor $(1 - x^2)^{-1/2}$ which appeared in the equation for the response time. The two integrals each contain a second factor which causes them to increase rapidly with x. Both integrals increase about 5 per cent for $x = 0.2$ and about 120 per cent when $x = 0.8$.

Logarithmic plots of A_1 and A_2 as functions of x and e_o are given in Fig. 5; the curves are straight lines with slopes of 3/2 and 5/2, respectively. Curves showing the variation of t_2, A_1, and A_2 with x for a fixed e_o are given in Fig. 6.

III. VISCOUS FRICTION

If motion of the output shaft is opposed by a viscous friction torque which is proportional to the velocity, new equations are needed to describe the variation of error with time and the shape of the torque-reversal curve must be changed for optimum performance.

System Equations

Considering only viscous friction and inertia, the equation for the response of the system to a step input is

$$J\ddot{e} + f\dot{e} = \pm T_m \qquad (28)$$

where f is the friction coefficient and the sign of the torque will again depend on e and $\dot{e}$.

The general solutions of this equation are

$$\dot{e} = \pm\Omega + K_1 e^{-t/\tau} \qquad (29)$$

$$e = \pm\Omega t - K_1\tau e^{-t/\tau} + K_2 \qquad (30)$$

where $\Omega = \frac{T_m}{f}$, the maximum or limiting velocity;

$\tau = \frac{J}{f}$, the time constant;

and K_1 and K_2 depend on the initial conditions.

For the same initial conditions as before, the error rate and error during the accelerating interval are

$$\dot{e} = -\Omega(1 - e^{-t/\tau}) \qquad (31)$$

$$e = e_o - \Omega t + \Omega\tau(1 - e^{-t/\tau}) \qquad (32)$$

Evidently the error rate increases exponentially toward the limiting value Ω, while the error is reduced, slowly at first and finally at a constant rate. As in the previous systems, it is proposed to reverse the torque at a time t_1 which is chosen in such a way that the error and error rate become zero simultaneously at time t_2. The complete response will then be as shown in Fig. 6A, which may be compared with Figs. 1(a) and 1(b).

At time t_1, the error has been reduced to

$$e(t_1) = e_o - \Omega t_1 + \rho\Omega\tau \qquad (33)$$

and the error rate is

$$\dot{e}(t_1) = -\rho\Omega \qquad (34)$$

where
$$\rho = 1 - e^{-t_1/\tau}, \tag{35}$$

and represents the fraction of the limiting speed which is attained by the output shaft at the switching point. Equations (33), (34), and (35) provide the initial conditions for the decelerating interval between t_1 and t_2.

During the deceleration interval, the equations become

$$\dot{e} = \Omega - (\rho + 1)\Omega e^{-(t - t_1)/\tau} \tag{36}$$

$$e = \Omega t - 2\Omega t_1 - \Omega\tau + (\rho + 1)\Omega\tau e^{-(t - t_1)/\tau} + e_o \tag{37}$$

Imposing the conditions $\dot{e}(t_2) = 0$, $e(t_2) = 0$, we obtain after some algebraic manipulation two equations

$$0 = \Omega - (\rho + 1)\Omega e^{-(t_2 - t_1)/\tau} \tag{38}$$

$$0 = \Omega(t_2 - t_1) - \Omega t_1 + e_o \tag{39}$$

which are analogous to Eqs. (18) and (19). Except for the nature of the equations, it would be possible to solve Eqs. (35), (38), and (39) to determine ρ, t_1, and t_2 for any particular e_o. It is much more convenient, however, to consider that t_1 is given. Using the value of ρ computed from Eq. (33), we find $(t_2 - t_1)$ from

$$e^{-(t_2 - t_1)/\tau} = \frac{1}{\rho + 1} \tag{40}$$

which is derived from Eq. (38). The response time is then

$$t_2 = t_1 + \overline{(t_2 - t_1)} \tag{41}$$

and the initial error is

$$e_o = 2\Omega t_1 - \Omega t_2 \tag{42}$$

Some typical response curves for a system of this type are given in Fig. 6B for $\tau = 0$ and $\tau = 1$. When τ is made zero by eliminating the inertia, switching occurs at $e = 0$; with no inertia, the output shaft rotates only when torque is applied and comes to rest immediately when the torque is removed. The switching point for $\tau = 1$ occurs at relatively small values of e, increasing to $e_s = 0.308$ for large initial errors.

Phase-plane plots are given in Fig. 6C.

Switching

The required torque-reversal curve in the phase-plane can be determined numerically, using Eqs. (33) and (34) which give the coordinates of the switching point for a particular value of e_o. A more direct method is obtained by recognizing that the switching boundary is also a particular trajectory of the system, passing through the origin. Noting that the torque is positive along this curve, we return to Ecs. (29) and (30), substitute the conditions $\dot{e}(0) = 0$ and $e(0) = 0$, and obtain

$$\dot{e}_s = \Omega (1 - e^{-t/\tau}) \tag{43}$$

$$e_s = \Omega t - \Omega\tau (1 - e^{-t/\tau}) \tag{44}$$

Eliminating t, we obtain

$$e_s = -\Omega\tau \ln\left[1 - \frac{\dot{e}_s}{\Omega}\right] - \tau\dot{e}_s \tag{45}$$

as the required relation between $\dot{e}_s$ and e_s. The boundary for the fourth quadrant is obtained by substituting negative values of $\dot{e}_s$; the boundary in the second quadrant is obtained from symmetry considerations.

Response Criteria

Inasmuch as no analytic expression for the response time is available, we will settle for a discussion of limiting cases and a graph obtained from numerical calculations.

If the initial error is small, the maximum velocity attained will also be small and the friction term will be relatively ineffective. The behavior of the system is then essentially that of the idealized system without friction. This expectation can be verified in two ways.

When $\dot{e}_s/\Omega$ is small enough, we can use the relation[18]

$$\ln(1 - x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$$

to reduce Eq. (45) to

$$e_s = \Omega\tau\left[\frac{\dot{e}_s}{\Omega} + \frac{\dot{e}_s^{\,2}}{2\Omega^2} + \frac{\dot{e}_s^{\,3}}{3\Omega^3} + \cdots\right] - \tau\dot{e}_s \tag{46}$$

$$\cong \frac{\tau}{2\Omega}\,\dot{e}_s^{\,2} \tag{47}$$

$$\cong \frac{J}{2T_m}\,\dot{e}_s^{\,2} \tag{48}$$

This equation is the same as Eq. (17) which described the torque-reversal curve of the optimum system discussed originally.

When the initial error is small, the response time of the system with viscous friction is essentially the same as the response time of the system without friction. When e_o is small, the maximum velocity attained is computed from

$$\rho \cong \frac{t_1}{\tau} \tag{49}$$

Using Eq. (40), we then have

$$e^{-(t_2 - t_1)/\tau} \cong \frac{1}{1 + t_1/\tau} \tag{51}$$

$$\cong 1 - \frac{t_1}{\tau} + \left[\frac{t_1}{\tau}\right]^2 \tag{52}$$

Expanding the exponential function in a series and taking only two terms, we get

$$\frac{t_2 - t_1}{\tau} \cong \frac{t_1}{\tau} - \left[\frac{t_1}{\tau}\right]^2$$

or

$$t_2 \cong 2\, t_1 - \frac{t_1^2}{\tau} \tag{53}$$

Substituting Eq. (53) into Eq. (42), we finally obtain

$$e_o \cong \frac{\Omega}{\tau}\, t_1^2 = \frac{T_m}{J}\, t_1^2 \tag{54}$$

or

$$t_1 \cong \sqrt{\frac{J}{T_m}\, e_o} \tag{55}$$

which is identical with Eq. (10). Since the response time is nearly $2\, t_1$, it follows that

$$t_2 \cong 2\sqrt{\frac{J}{T_m}\, e_o} \tag{56}$$

as before.

If the initial error is large, it is convenient to write Eq. (42) in the form

$$t_2 = \frac{e_o}{\Omega} + 2\,(t_2 - t_1) \tag{57}$$

since the limiting value of $(t_2 - t_1)$ can be found from Eq. (40) by taking $\rho = 1$. This substitution gives

$$e^{-(t_2 - t_1)/\tau} = \frac{1}{2} \tag{58}$$

or

$$(t_2 - t_1) = 0.693\,\tau \tag{59}$$

For large initial errors, the response time therefore approaches

$$t_2 = \frac{e_o}{\Omega} + 1.386\,\tau \tag{60}$$

The relations are summarized in Fig. 6D which shows the response time as a function of initial error. A curve for the optimum system without friction is given for comparison. As predicted, the curve for $\tau = 1$ coincides with the curve for the ideal system for initial errors less than about one-half and is a linear function of initial error when e_o is greater than two. A system with no inertia $(\tau = 0)$ is superior to the ideal system for initial errors less than four. This result may be expected from the fact that the inertialess system reaches its maximum velocity immediately and wastes no time in acceleration; for initial errors greater than four, the greater maximum velocities which are possible in the ideal system result in a smaller response time.

Curves of integrated absolute and squared error as functions of the initial error, plotted using logarithmic coordinates, are given in Fig. 6E. The dashed curves are for the optimum system without friction. For $\tau = 0$, the curves are computed from the equations

$$t_1 = t_2 = \frac{1}{\Omega} e_o \tag{61}$$

$$A_1 = \frac{1}{2\Omega} e_o^2 \tag{62}$$

$$A_2 = \frac{1}{3\Omega} e_o^3 \tag{63}$$

For $\tau = 1$, the curves were obtained by numerical integration from the computed response curves of Fig. 6B.

The integrated absolute or squared error for the system with friction and inertia is always greater than the corresponding integral for either (1) a system with inertia and no friction (the optimum system originally considered) or (2) a system with friction and no inertia ($\tau = 0$). For small initial errors, the curves for the system with $\tau = 1$ approach the curves for the original optimum system without friction; for large initial errors, the curves are asymptotic to the curves for $\tau = 0$, as may be expected from the nature of the response. These limiting curves evidently supply lower bounds for the integrals and indicate the best possible performance to be expected from a given motor. Any method of control other than the optimum switching methods will result in poorer performance, shown by an increase in the integrated absolute or squared error for any step disturbance.

Although the special values $T_m = J = f = 1$ were used to compute the response and phase-plane curves presented in this section of the report, the curves for $\tau = 1$ may be treated as functions of dimensionless variables. To apply these results to other systems, it is only necessary to consider that time has been measured relative to τ and error relative to $\Omega\tau$. With these scale changes, the curves apply to any similar system.

<u>Combined Coulomb and Viscous Friction</u>

If the proposed optimum system is subjected to small coulomb <u>or</u> viscous friction torques, redesign of the nonlinear network will result in the desired type of operation with a moderate sacrifice in performance, as has been indicated. No studies of the effects of combined coulomb <u>and</u> viscous friction have been made, but additional degradation of the response is to be expected.

Effects of simultaneous coulomb and viscous friction can be minimized, however, by a suitable selection of the torque-reversal curve and the corresponding nonlinear network. As already noted[12], the torque-reversal curve must be a trajectory of the system for the deceleration condition, passing through the origin. For this case, the required curve can be computed by substituting

$$\Omega' = \frac{T_m + T_f}{f} \tag{64}$$

$$= \Omega\,(1 + x) \tag{65}$$

for Ω in Eq. (45).

IV. SWITCHING ERRORS

Hopkin[11] describes two methods for converting the theoretical torque-reversal curve into a physical switching device. In one method the phase-plane plot is reproduced on the face of a cathode-ray tube in which the horizontal and vertical deflections of the beam depend directly on the error and error rate. An opaque mask covers the region on one side of the torque-reversal curve, and a photocell determines from the location of the spot whether positive or negative torque is required. This scheme, which permits any required torque-reversal curve to be easily realized, is bulky and complicated.

More commonly[10-12], switching is carried out physically by comparison of two voltages, one proportional to the error and the other proportional to a nonlinear function of the error rate*. The nonlinear function is generated by biased-diode networks whose design is based on Eq. (17).

In previous investigations this network has been made to produce a close approximation to the desired switching procedure, as determined by direct measurement and also by observation of the overall transient behavior of the system. No detailed study has been made of the required accuracy of this approximation. Since a rather poor approximation might

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*The reverse process, using a nonlinear function of error and a voltage proportional to error rate, has also been used[12].

give satisfactory results and be much easier to realize, a study of the effects of improper switching was made.

For simplicity, friction torques are ignored and switching is assumed to be instantaneous. The system considered is the system proposed by McDonald and Hopkin, except that the torque-reversal curve does not coincide with the correct curve given by

$$\dot{e}_s^{\,2} = \frac{2\,T_m}{J}\,e_s \qquad (17)$$

Equation (17), which gives the torque-reversal curve in the fourth quadrant of the phase plane, may be written

$$e_s = -\frac{J}{2\,T_m}\,\dot{e}_s\left|\dot{e}_s\right| \qquad (66)$$

to include both the second and fourth quadrants. The torque-reversal curves which have been studied are generalizations of Eq. (66) and are expressed as

$$e_s = -\,k\,\dot{e}_s\left|\dot{e}_s\right|^{\alpha - 1} \qquad (67)$$

The coefficient "k" may be said to control the "location" of the torque-reversal curve, while the exponent "α" governs the "shape" of the curve. The correct values, given by Eq. (66), are $k = J/2T_m$ and $\alpha = 2$. For convenience, "location" and "shape" errors are considered separately.

V. BOUNDARY LOCATION

System Equations

To examine the effects of varying k, it is assumed that $\alpha = 2$. The torque-reversal curve in the fourth quadrant is therefore

$$e_s = k \dot{e}_s^2 \qquad (68)$$

Letting $T_m = J = 1$, as before, the correct coefficient is $k = 0.5$. If the actual k is greater than the correct value, the torque-reversal curve moves closer to the horizontal axis of the phase plane and "early" switching results. Smaller values of k move the curve toward the vertical axis and cause "late" switching, with consequent overshoot or oscillation.

The discussion of the switching process is simplified by use of a parameter r, defined by

$$r = \frac{e_1}{e_0} , \qquad (69)$$

where e_1 is the error at the time t_1 when the first switching takes place. The parameter r is a function of k but does not depend on e_0. Early switching corresponds to $r > 0.5$, while late switching means $r < 0.5$; the two cases require different treatment.

In all cases, the error is given by

$$e = e_0 - \frac{\dot{e}^2}{2} \qquad (70)$$

for $0 < t < t_1$. For $r > 0.5$, the point in the phase plane then follows the torque-reversal curve; this behavior may be understood by reference to Fig. 7.

The dashed curve in Fig. 7 is the torque-reversal curve for $r = 0.7$. In Fig. 7(a), the point has been allowed to travel past this curve to the curve for $r = 0.6$ before the torque is reversed. The point then moves on a trajectory which would result in an error equal to $0.2\ e_o$ at the point of zero error rate. The torque is reversed again, however, at a curve corresponding to $r = 0.8$, and the output shaft is again accelerated. The cycle of acceleration and deceleration is repeated until the point reaches the origin.

In Fig. 7(b), switching occurs on curves for which $r = 0.66$ and 0.74. The number of oscillations about the torque-reversal curve ($r = 0.7$) has increased and the decrease in error between switching operations has been reduced.

The switching procedure described is artificial and no attempt is made to interpret it physically; it is used to provide an easily defined limiting process which will explain system operation for ideal, instantaneous switching. In the limit, switching takes place at the curves $r = 0.7 \pm \epsilon$, with ϵ approaching zero; the point then makes a large number of small oscillations about the torque-reversal curve. In effect, the point follows the torque-reversal curve, along which e and $\dot{e}$ are given by Eq. (68). (Needless to say, this behavior is physically impossible and even a close approximation would impose severe demands on the relay.)

For $r < 0.5$, the error is given by Eq. (70) until the point reaches the torque-reversal curve. The trajectory then becomes

$$e = e_2 + \frac{\dot{e}^2}{2} \tag{71}$$

which applies until the next intersection with the torque-reversal curve, at time t_3. Here e_2 denotes the error at time t_2 when the error rate is zero; the value of e_2 is

$$\begin{aligned} e_2 &= e_o - 2(1 - r)\, e_o \\ &= e_o\,(2r - 1) \end{aligned} \tag{72}$$

At the next switching point,

$$e_3 = r\, e_2 \tag{73}$$

and it may be concluded that, in general,

$$r = \frac{e_1}{e_o} = \frac{e_3}{e_2} = \frac{e_5}{e_4} = \frac{e_7}{e_6} = \cdots\cdots \tag{74}$$

$$(2r - 1) = \frac{e_2}{e_o} = \frac{e_4}{e_2} = \frac{e_6}{e_4} = \frac{e_8}{e_6} = \cdots\cdots \tag{75}$$

Equations (74) and (75) furnish the error at every switching point and every crossing of the horizontal axis. The trajectories joining these points are all parts of a parabolic curve along which e varies as $\dot{e}^2/2$. This curve can be plotted once on a separate sheet of graph paper and traced to obtain the phase-plane plots.

Phase-plane plots for r = 0.2, 0.3, 0.4, 0.6, 0.7, and 0.8 are given in Fig. 8; the plot for r = 0.5 is found in Fig. 2.

Response Criteria

Curves showing error as a function of time, based on the phase-plane plots, are given in Fig. 9 for a unit initial error and r = 0.7 and 0.3. As would be expected from the phase-plane plots, the error decreases slowly and monotonically for $r > 0.5$, resulting in an increased response time. For $r < 0.5$, the error decreases to zero in a series of oscillations of diminishing period.

The integrated absolute and squared error was determined numerically from curves similar to those of Fig. 9; the results are given in Fig. 10 as functions of k and r. The relation between k and r may be derived in the following way. At the first switching point,

$$e_1 = r\, e_o = k\, \dot{e}_1^{\,2} \tag{76}$$

from Eqs. (68) and (69). The error rate at this point is also given by

$$\dot{e}_1^{\,2} = 2\,(1 - r)\, e_o \tag{77}$$

from Eqs. (69) and (70). Eliminating $\dot{e}_1$, we obtain

$$k = \frac{r}{2\,(1 - r)} \tag{78}$$

or

$$r = \frac{2\,k}{1 + 2\,k} \tag{79}$$

A linear scale was used for k, since it is likely to be the constant which is changed in adjusting the system.

Unlike the error integrals, the response time is easily computed and expressed in closed form. From Eq. (5), the time required to reach the first switching point is

$$t_1 = \sqrt{2\,(1 - r)\,e_o} \tag{80}$$

For $r \leqslant 0.5$, the time required to next reach zero error rate, which might be considered as the time for a half cycle of the oscillation, is

$$t_2 = 2\,t_1 = 2^{3/2}\sqrt{(1 - r)\;e_o} \tag{81}$$

By analogy, the time required for subsequent half cycles will be

$$t_4 - t_2 = 2^{3/2}\sqrt{(1 - r)\;|e_2|} \tag{82}$$

$$t_6 - t_4 = 2^{3/2}\sqrt{(1 - r)\;e_4} \tag{83}$$

$$t_8 - t_6 = 2^{3/2}\sqrt{(1 - r)\;|e_6|} \tag{84}$$

and so forth. The absolute value marks are introduced to avoid difficulty with signs, since e_2, e_6, e_{10}, $\cdots$ are negative. The response time is the sum of an infinite number of such intervals, or

$$\text{Response Time} = 2^{3/2}\sqrt{1 - r}\left[\sqrt{e_o} + \sqrt{|e_2|} + \sqrt{e_4} + \sqrt{|e_6|} + \cdots\right] \tag{85}$$

However, from Eq. (75), we have

$$e_2 = (2r - 1)\;e_o \tag{86}$$

$$e_4 = (2r - 1)\;e_2 = (2r - 1)^2\;e_o \tag{87}$$

$$e_6 = (2r - 1)\;e_4 = (2r - 1)^3\;e_o \tag{88}$$

and so forth. Expressing Eq. (85) entirely in terms of e_o, and writing $(1 - 2r)$ instead of $(2r - 1)$ in order to account for the absolute value marks, the response time is found to be

$$\text{Response Time} = 2^{3/2} \sqrt{(1 - r)\ e_o} \left[1 + z + z^2 + z^3 + \cdots \right] \tag{89}$$

$$\text{where} \quad z = (1 - 2r)^{1/2} \tag{90}$$

This series is an infinite geometric progression, whose sum is[18]

$$\text{Response Time} = 2^{3/2} \frac{\sqrt{1 - r}}{1 - \sqrt{1 - 2r}} \sqrt{e_o} \tag{91}$$

provided that $0 < r \leqslant 0.5$.

For $0.5 \leqslant r < 1.0$, a different approach is used, based on the relation

$$\text{Response Time} = t_1 + \int_{e_1}^{0} \frac{de}{\dot{e}} \tag{92}$$

The error rate along the torque-reversal curve may, by means of Eq. (68), be expressed as

$$\dot{e} = -\sqrt{\frac{e}{k}} \tag{93}$$

with the result that

$$\text{Response Time} = t_1 + 2\sqrt{k\ e_1} \tag{94}$$

Equation (94) reduces, using Eqs. (80), (78), and (69), to

$$\text{Response Time} = \sqrt{\frac{2}{1 - r}} \sqrt{e_o} \tag{95}$$

Equations (91) and (95) give the response time for any value of r between zero and unity, Eq. (91) being used for cases of late switching and Eq. (95) for early switching. Values computed from these equations are given in Fig. 11. Also shown in this figure are other values of time at which the error is zero, taken from curves similar to those of Fig. 9.

The response time is finite except for $r = 0$, which means a sustained oscillation, and $r = 1$, which corresponds to no reduction of initial error whatever. The response time increases rapidly as r is reduced below one-half, the correct value, and increases slowly with increasing r. On the basis of these observations, any error in the adjustment of the nonlinear lead network should be made in the direction of early switching; this conclusion would be strengthened by consideration of relay operating time, omitted in this study.

Since the nature of the response does not depend on the initial error, the curves shown in Figs. 10 and 11 may be corrected by the use of Eqs. (12), (13), and (14) for other values of J, T_m or e_o. The dependence of response time, integrated absolute error, and integrated squared error on J/T_m and e_o is not affected by errors in k.

VI. BOUNDARY SHAPE

Given a biased-diode network which provides a voltage proportional to $\dot{e}^2$, it should be relatively easy to adjust the constant of proportionality to obtain the correct torque-reversal curve and the desired switching behavior. The real difficulty is the design of a network to produce an accurate square-law relationship between input error rate and output voltage.

In the following discussion, the torque-reversal curve is therefore taken as

$$e_s = -\frac{1}{2}\,\dot{e}_s\left|\dot{e}_s\right|^{\alpha-1} \qquad (96)$$

in general, or

$$e_s = +\frac{1}{2}\left|\dot{e}_s\right|^{\alpha} \qquad (97)$$

in the fourth quadrant. If α is 1 or 3, the cases which are selected for study, the absolute value marks in Eq. (97) can be omitted and a minus sign substituted for the plus sign. The coefficient, 1/2, has been chosen to make the torque-reversal curve pass through the point (1/2, -1) as the correct curve does; the curve does not coincide with the correct curve for other values of e and $\dot{e}$. As before, the special case $T_m = J = 1$ is considered.

The analytical procedure adopted is essentially graphical; the torque-reversal curve is plotted directly from Eq. (97) and representative trajectories are traced, using curves plotted from Eqs. (70) and (71). Some typical phase-plane plots are given in Figs. 12, 13, and 14.

System Equations

For $\alpha = 1$, the switching network is effectively a linear lead network whose properties have been studied[3,5]. The point starts out along a trajectory given by Eq. (70) and intersects the torque-reversal curve (now a straight line) for the first time at values of e and $\dot{e}$ obtained by solving simultaneously

$$e_1 = e_o - \frac{\dot{e}_1^{\,2}}{2} \tag{98}$$

from Eq. (70) and

$$e_1 = - \frac{\dot{e}_1}{2} \tag{99}$$

from Eq. (97). The error at the first switching point is

$$e_1 = \frac{-1 + \sqrt{1 + 8e_o}}{4} \tag{100}$$

Several types of response may be distinguished, depending on the initial error:

(a) If $e_o \leq 3/8$, the point follows the torque-reversal curve to the origin after its intersection with the curve at time t_1. Since the error rate is directly proportional to the error along the torque-reversal curve, the error decreases exponentially with time.

(b) If $3/8 < e_o < 1$, the point leaves the curve along a trajectory which could be expressed by Eq. (71), with e_2 positive, intersecting the torque-reversal curve again in the fourth quadrant; it then follows the torque-reversal curve to the origin.

(c) If $e_o = 1$, the point leaves the torque-reversal curve along the trajectory

$$e = \frac{\dot{e}^2}{2} \qquad (101)$$

which is a particular case of Eq. (71) and reaches the origin without further switching and in a finite time.

(d) If $e_o > 1$, the point leaves the torque-reversal curve along trajectories given by Eq. (71) and arrives at zero error rate with an error equal to

$$e_2 = -e_o - \frac{1}{2} + \frac{\sqrt{1 + 8e_o}}{2} \qquad (102)$$

This error may be taken as a new initial error and tests (a)-(d) again applied.

With a linear lead network, therefore, large initial errors are reduced in an oscillatory fashion at first, followed by an exponential decrease to zero; small initial errors will result in a monotonic response.

For $\alpha = 3$, the torque-reversal curve is closer to the horizontal axis than the correct curve for large errors. After the first intersection of the trajectory with the torque-reversal curve, two types of behavior may be noted:

(a) For $e_o > 10/27$, the trajectories coincide with the torque-reversal curve until the error is $4/27$; at this point a series of oscillations begins.

(b) For $e_o < 10/27$, all trajectories reach the origin as the result of a series of oscillations.

In this case, large initial errors are reduced in a monotonic fashion at first, followed by an oscillatory decrease to zero.

The critical initial error, 10/27, is obtained in the following way. Consider a point on the torque-reversal curve; draw through this point a trajectory corresponding to a decelerating torque. If motion along this trajectory would cause the point to immediately enter the region of accelerating torque, the point will abandon this trajectory and follow the torque-reversal curve, as discussed earlier (p. 2[illegible]). If motion along this trajectory causes the point to remain in a region of decelerating torque, the point will follow the trajectory until its next intersection with the torque-reversal curve. A special situation is evidently obtained if the trajectory coincides locally with the torque-reversal curve, that is, if the rate of change of e with $\dot{e}$ along the trajectory is the same as the rate of change of e_s with $\dot{e}_s$ along the torque-reversal curve. Stated mathematically, this situation requires

$$\left.\frac{de}{d\dot{e}}\right|_{\dot{e}_s} = \frac{de_s}{d\dot{e}_s} \tag{103}$$

or

$$\left.\frac{d\dot{e}}{de}\right|_{e_s} = \frac{d\dot{e}_s}{de_s} \tag{104}$$

For $\alpha = 3$, the torque-reversal curve is

$$e_s = -\frac{1}{2} \dot{e}_s^{\,3} \; ; \tag{105}$$

a typical decelerating trajectory is

$$e = e_2 + \frac{\dot{e}^2}{2} \; . \tag{71}$$

Applying Eq. (103) and specifying that $\dot{e}$ in Eq. (71) be taken at a point on the torque-reversal curve, we obtain

$$-\frac{3}{2}\,\dot{e}_s^{\,2} = \dot{e}_s \tag{106}$$

or

$$\dot{e}_s = -\frac{2}{3} \tag{107}$$

At this point, the error is

$$e_s = -\frac{1}{2}\left(-\frac{2}{3}\right)^3$$

$$= \frac{4}{27} \tag{108}$$

From Eq. (70), the corresponding initial error is found from

$$\frac{4}{27} = e_o - \frac{1}{2}\left(-\frac{2}{3}\right)^2 , \tag{109}$$

which gives

$$e_o = \frac{10}{27} . \tag{110}$$

This analysis serves to determine exactly a critical initial error whose approximate magnitude is easily obtained by use of graphical methods.

Response Criteria

The phase-plane plots furnish a complete description of the system response and can be used to derive curves of error as a function of time to which the various response criteria can be applied. While it would be possible to construct these curves entirely from the phase-plane plots, it is probably more satisfactory to obtain only certain crucial facts from the phase-plane: the error and error rate where trajectories arrive at, or depart from, the torque-reversal curve, and the error at points on the trajectories for which the error rate is zero. This data may be used connection with equations for error as a function of time to obtain the desired curves.

When the trajectories do not coincide with the torque-reversal curve, the error can be calculated from Eq. (3), using appropriate initial conditions. For $\alpha = 1$, the time variation of the error is given by the differential equation

$$\frac{1}{2}\dot{e} + e = 0 \qquad (111)$$

when the point describing the system is following the torque-reversal curve. This equation has the solution

$$e = K\, e^{-2(t - t_k)} \qquad (112)$$

where K is the error at time t_k when this solution is first applicable; this solution applies from the time t_k to infinity, when the error is zero.

For $\alpha = 3$, the time variation of the error along the torque-reversal curve can be found from

$$dt = \frac{de}{\dot{e}} \qquad (113)$$

and the equation, obtained from Eq. (97),

$$\dot{e} = -\sqrt[3]{2\,e} \qquad (114)$$

The resulting equation for error is

$$t - t_1 = \frac{3}{2^{4/3}} \left[e_1^{2/3} - e^{2/3} \right] \qquad (115)$$

where e_1 is again the error at time t_1 when the first switching occurs; this equation applies until the error is reduced to 4/27. It may be noted that the error values to be substituted in this equation are less than e_1.

Typical curves of error as a function of time for a particular initial error are given in Fig. 15. For $\alpha = 1$, there is one overshoot, followed by an exponential decrease to zero; for $\alpha = 3$, the error decreases to zero in a series of oscillations of diminishing period. Response times estimated from a family of similar curves are given in Fig. 15A for both $\alpha = 3$ and the ideal system ($\alpha = 2$).

A system with a linear load network ($\alpha = 1$) has the very interesting property that its response time is finite for a discrete set of initial errors ($e_0 = 1, 3, 5, \cdots$) and infinite, theoretically speaking, for all others. Practically, of course, the error is negligibly small in a reasonable time; the meaning of "negligible", however, is a subjective matter and therefore a curve of response time as a function of initial error for this case is not very meaningful. For this reason the curve is not presented.

No difficulty arises in the determination of the integrated absolute or squared error. Values of these integrals, obtained numerically from the response curves, are given in Fig. 16 as functions of α with the initial error as a parameter. While $\alpha = 2$ is clearly the optimum value, the integrated errors are relatively independent of α for small initial errors.

VII. CONCLUSIONS

General

Moderate deviations of the system parameters from their optimum values can be permitted without much sacrifice in the performance of the system, as measured by the response time, integrated absolute error, or integrated squared error for a step input.

Friction

A surprisingly large Coulomb friction torque, up to 50 per cent, can be tolerated if a simple adjustment is made in the switching device. Viscous friction is not significant for small errors, that is, errors which can be reduced to zero without reaching the maximum available speed, if a similar adjustment is made.

Switching

The torque-reversal curve should be designed to have an approximately "parabolic" shape and the correct location in the phase plane; imperfections in the switching device are less serious if they result in "early" switching.

Response Criteria

The time required for the error to become zero after a step disturbance is a satisfactory measure of system quality, provided the time is finite. If the time is not finite, the integrated absolute or squared error provides an objective criterion. Since the integrated absolute error is easier to calculate and is minimized by the same parameter values which minimize the integrated squared error, its use is recommended[16].

REFERENCES

1. H. L. Hazen, "Theory of Servomechanisms". Jour. Frank. Inst., Vol. 218, September, 1934, pp. 279-330

2. L. A. MacColl, Fundamental Theory of Servomechanisms. D. VanNostrand Company, New York, 1945, pp. 107-125

3. H. K. Weiss, "Analysis of Relay Servomechanisms". Jour. Inst. Aero. Sci., Vol. 13, July, 1946, pp. 364-376

4. D. A. Kahn, "An Analysis of Relay Servomechanisms". Trans. A.I.E.E., Vol. 68, 1949, pp. 1079-1087

5. R. J. Kochenburger, "A Frequency-Response Method for Analyzing and Synthesizing Contactor Servomechanisms". Trans. A.I.E.E., Vol. 69, 1950, pp. 270-283

6. T. A. Rogers and W. C. Hurty, "Relay Servomechanisms: The Shunt-Motor Servo with Inertia Load". Trans. A.S.M.E., Vol. 72, November, 1950, pp. 1163-1172

7. J. R. Dutilh, "Theory of Relay Servomechanisms". Onde Elect., Vol. 30, October, 1950, pp. 438-445

8. D. McDonald, "Nonlinear Techniques for Improving Servomechanism Performance". Proc. Nat. Elect. Conf., Vol. 6, Chicago, 1950, pp. 400-421

9. ------------, "Multiple Mode Operation of Servomechanisms". Rev. Sci. Inst., Vol. 23, January, 1952, pp. 22-30

10. ------------, "Basic Research in Nonlinear Mechanics as Applied to Servomechanisms". Cook Research Laboratories Technical Report No. T16-1, Chicago, January, 1952

11. A. M. Hopkin, "A Phase-Plane Approach to the Compensation of Saturating Servomechanisms". Trans. A.I.E.E., Vol. 70, 1951, pp. 631-639

12. A. M. Uttley and P. H. Hammond, "The Stabilization of On-Off Controlled Servomechanisms". Automatic and Manual Control, Academic Press, New York, 1952, pp. 285-307. Discussions by R. W. Jones, J. C. West, D. T. Broadbent, J. P. Corbett, T. Persson, K. V. Diprose, and K. H. Doetsch.

13. E. L. Ince, Ordinary Differential Equations. Dover, New York, p. 69

14. G. S. Brown and D. P. Campbell, Principles of Servo-Mechanisms. John Wiley and Sons, New York, 1948

15. J. T. Carleton, "The Transient Behavior of the Two-Stage Rototrol Main Exciter Voltage Regulating System as Determined by Electrical Analogy". Trans. A.I.E.E., Vol. 68, 1949, pp. 59-63

16. F. C. Fickeisen and T. M. Stout, "Analogue Methods for Optimum Servomechanism Design". Trans. AIEE, Vol. 71, 1952

17. R. R. Caldwell and V. C. Rideout, "A Differential Analyzer Study of Certain Nonlinearly-Damped Servo-mechanisms". AIEE Technical Paper 53-107, 1953.

18. H. B. Dwight, Tables of Integrals and Other Mathematical Data. Macmillan Company, New York, 1947

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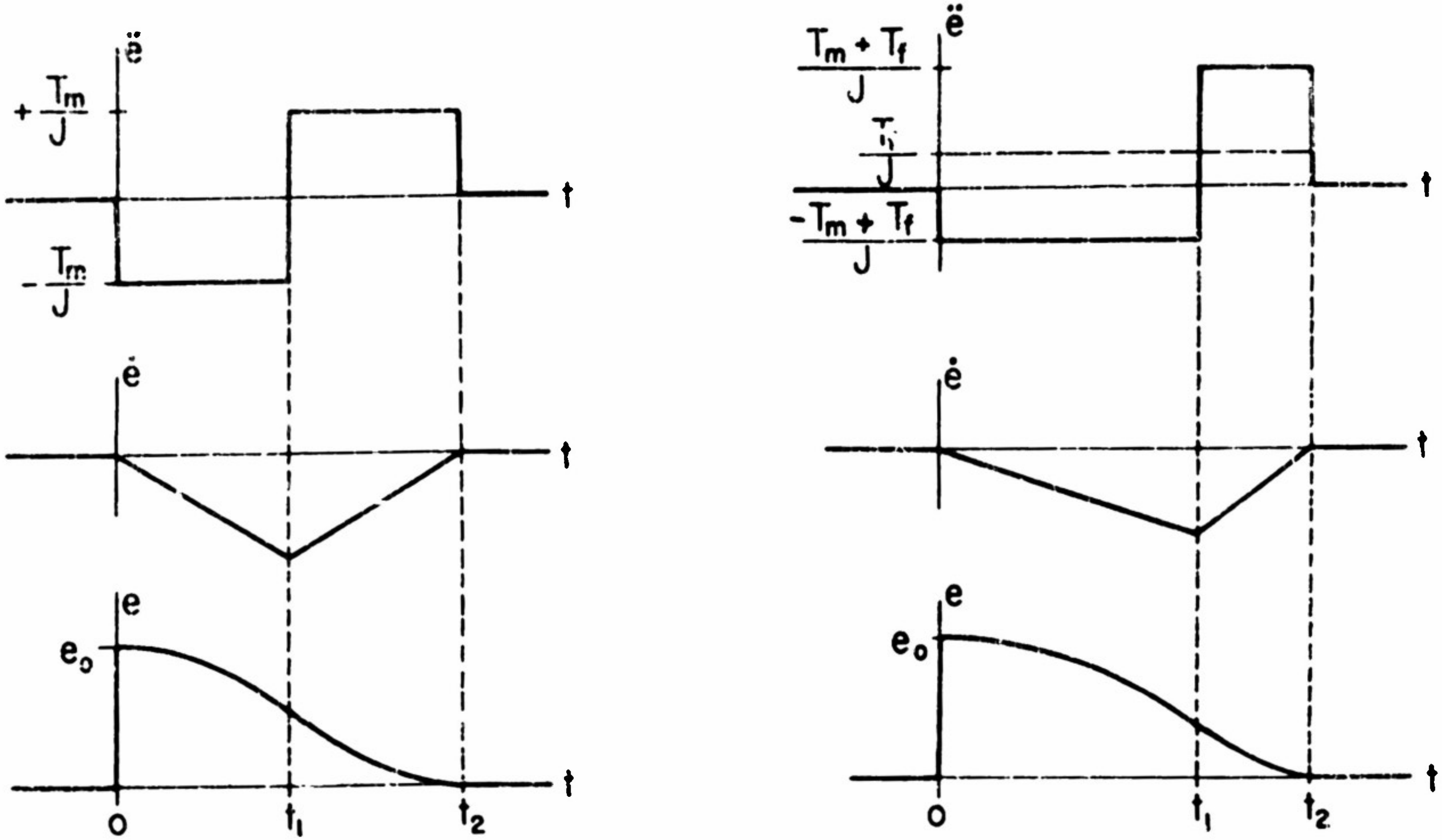
16. Integrated absolute error (A_1) and squared error (A_2) as functions of α for several values of initial error.



(a) IDEAL CASE

(b) WITH COULOMB FRICTION

FIGURE 1

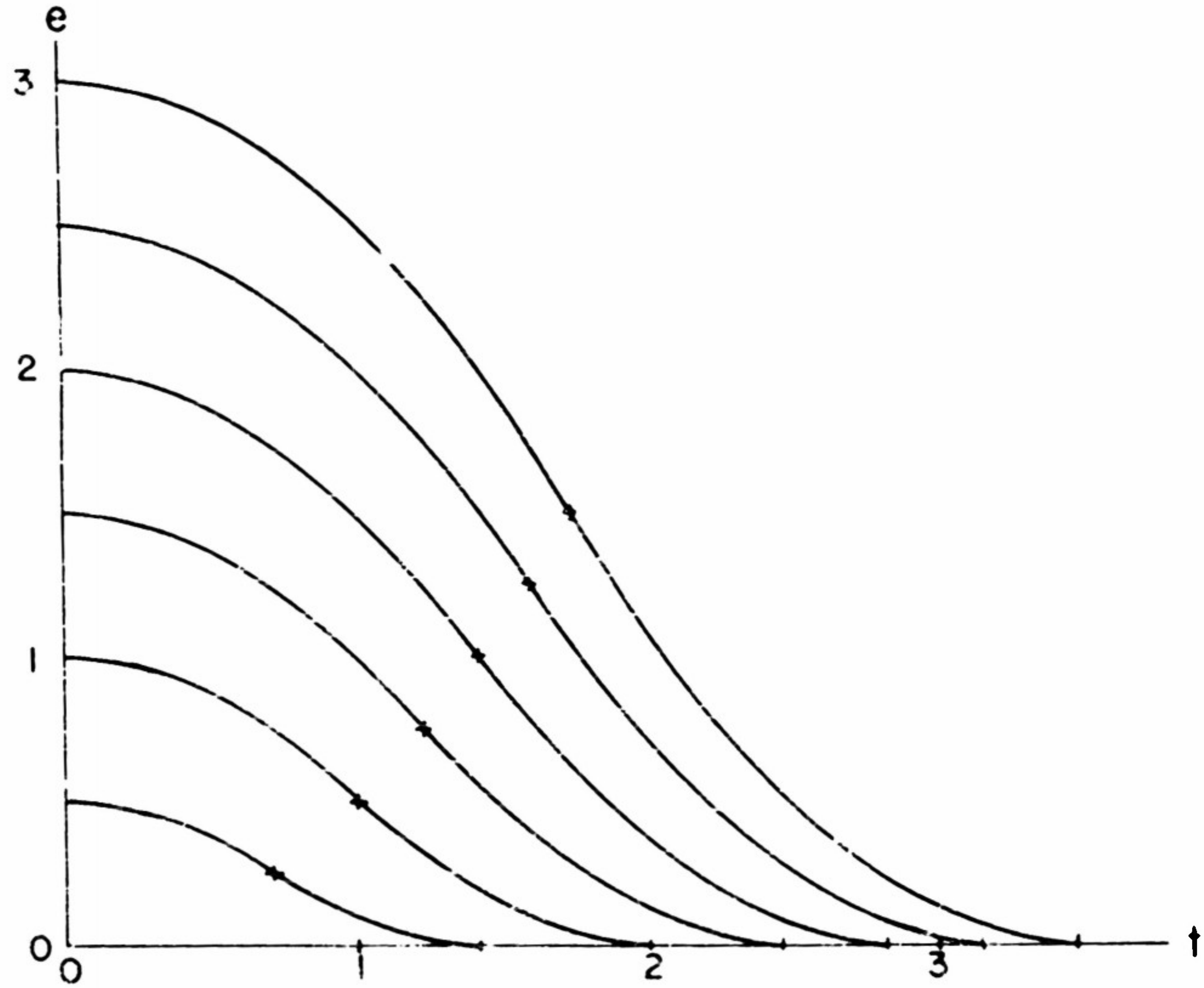


FIGURE 1(c)

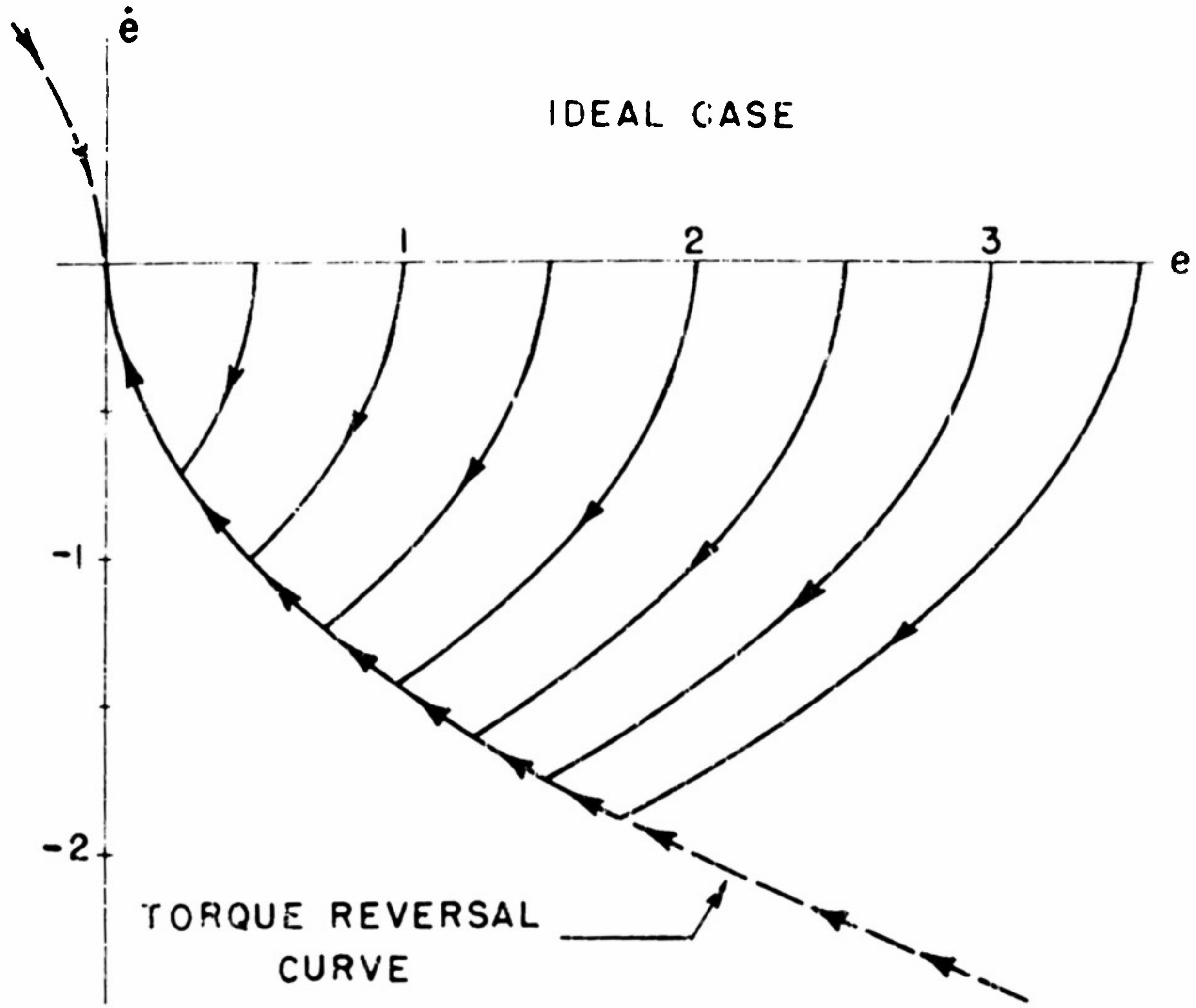


FIGURE 2

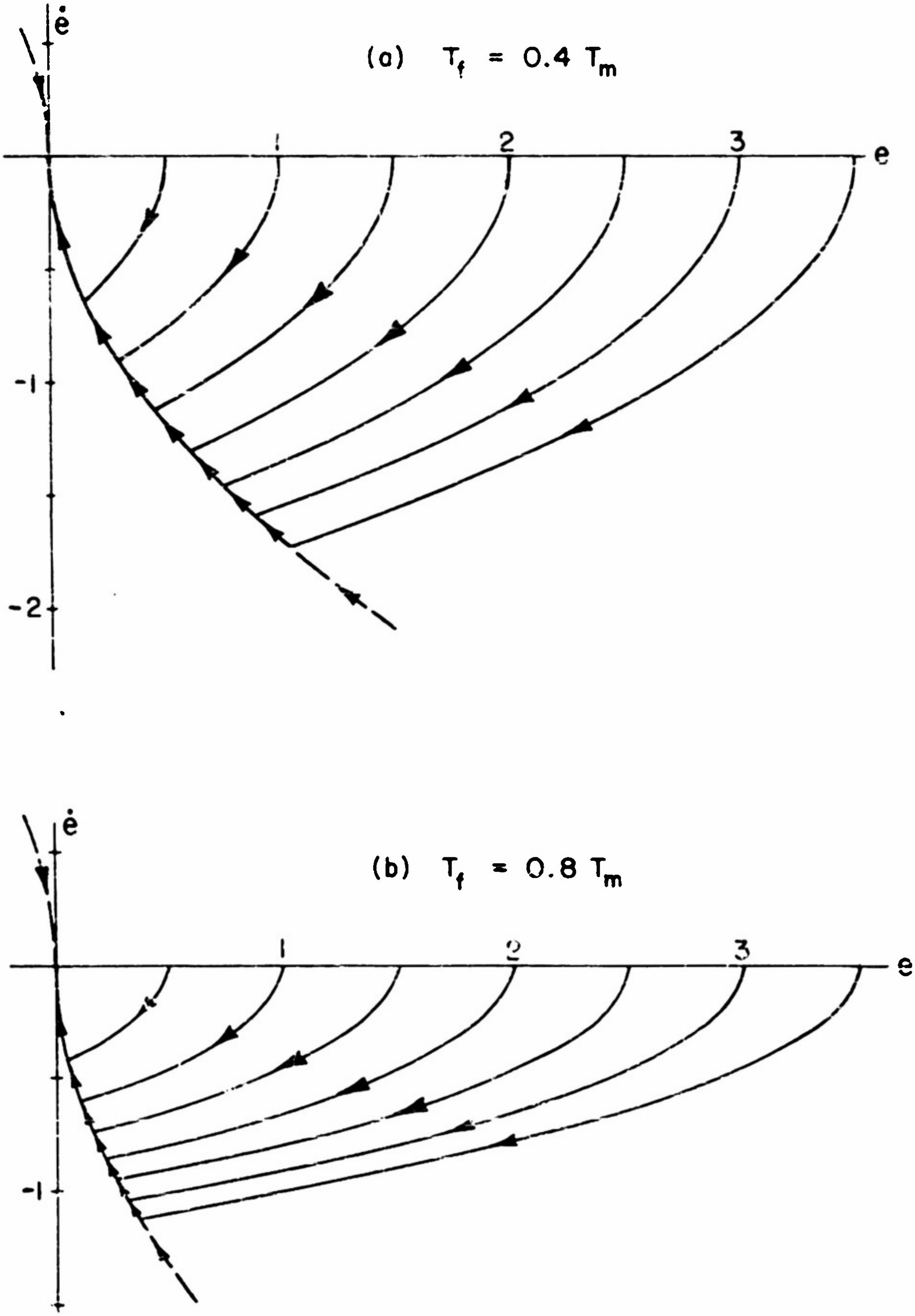


FIGURE 3

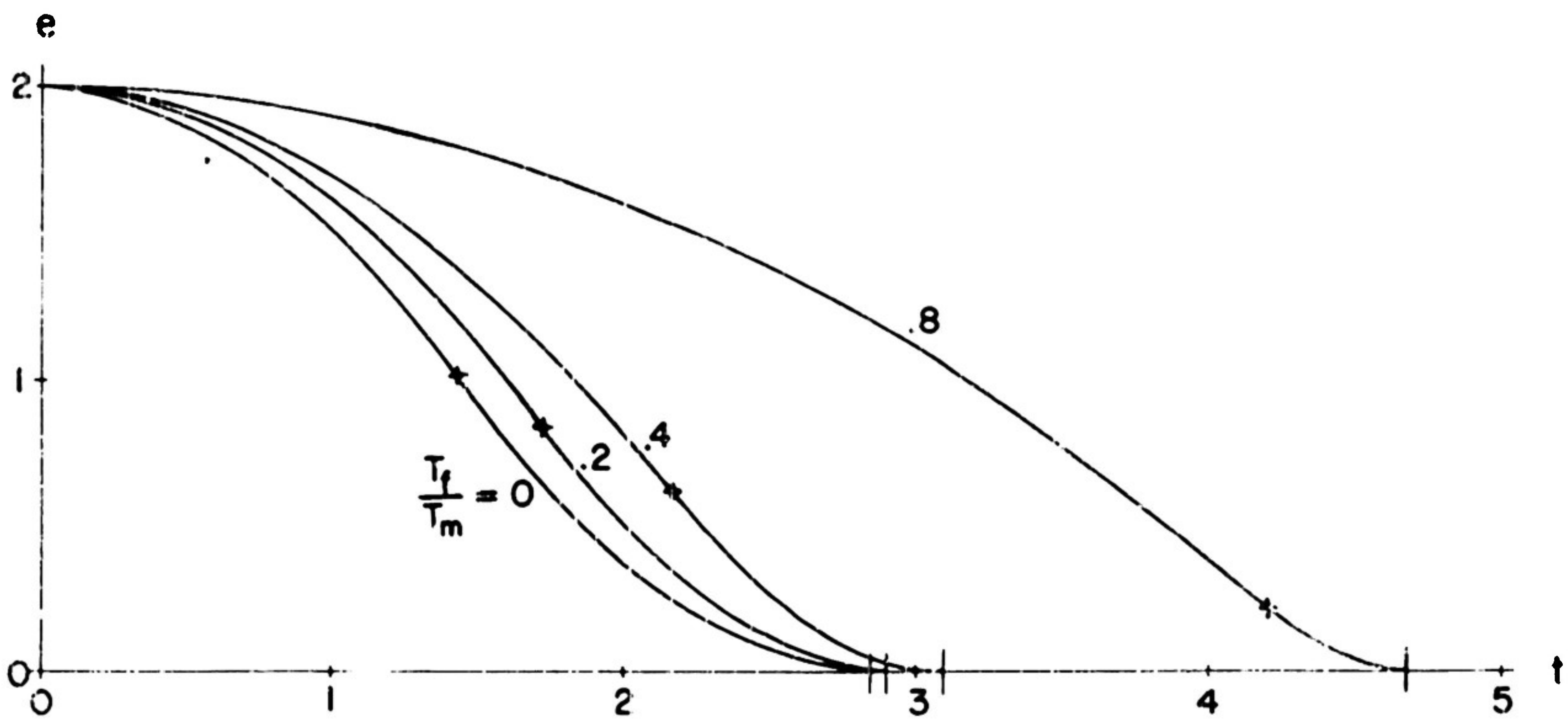


FIGURE 4

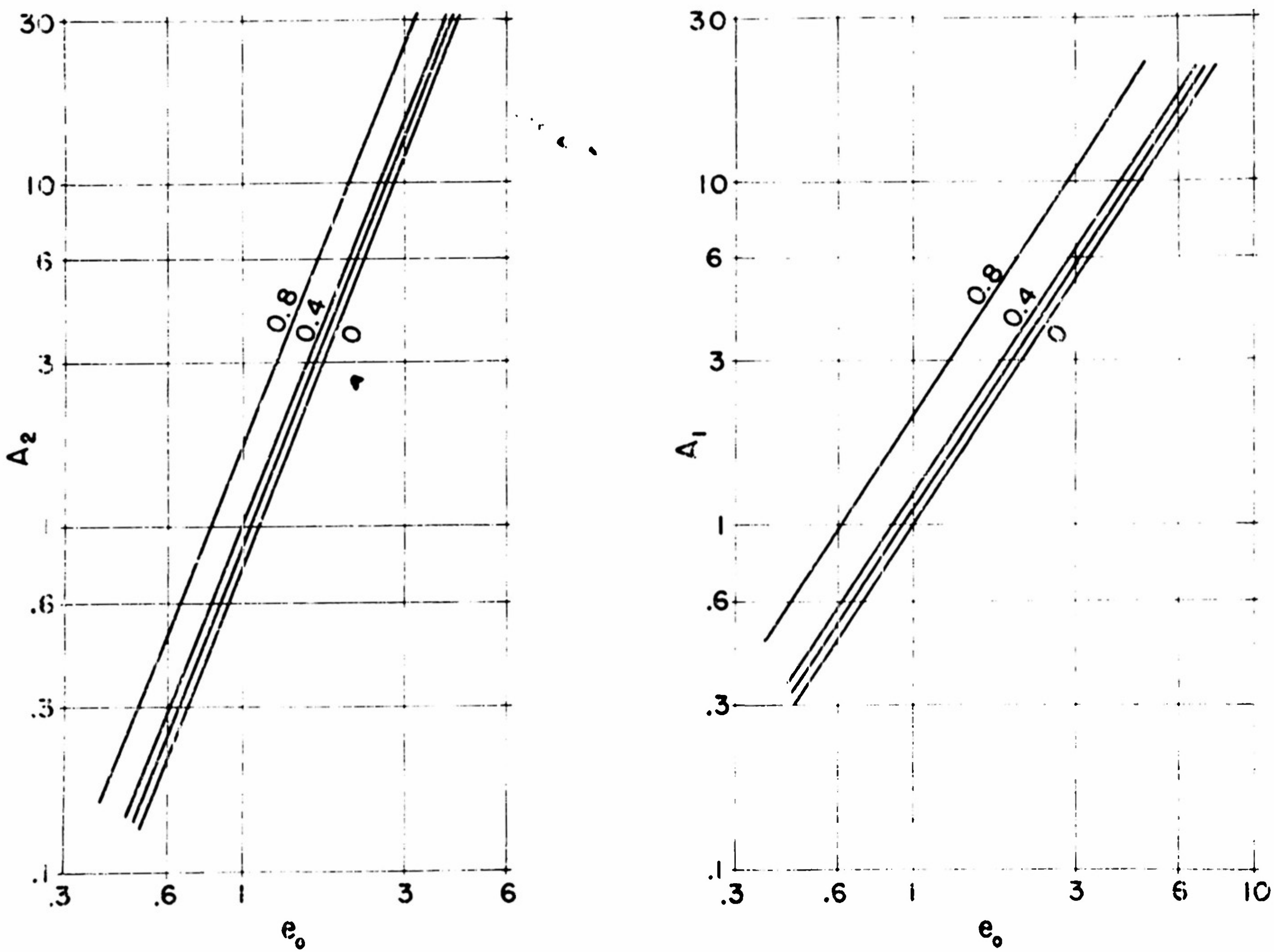

30
10
6
3
1
.6
.3
.1
A_2
0.8
0.4
0
.3
.6
1
3
6
e_o
30
10
6
3
1
.6
.3
.1
A_1
0.8
0.4
0
.3
.6
1
3
6
10
e_o

FIGURE 5

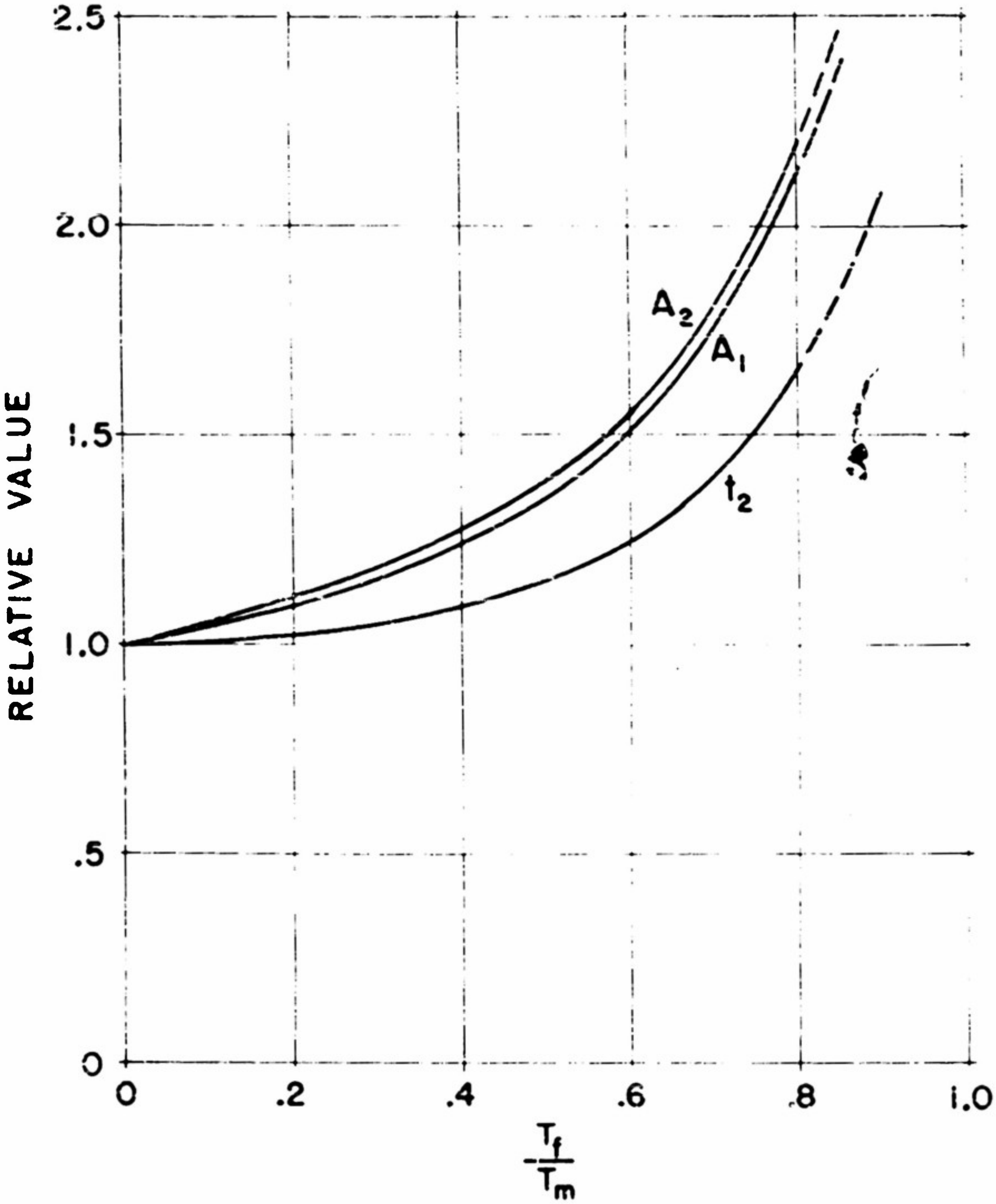


FIGURE 6

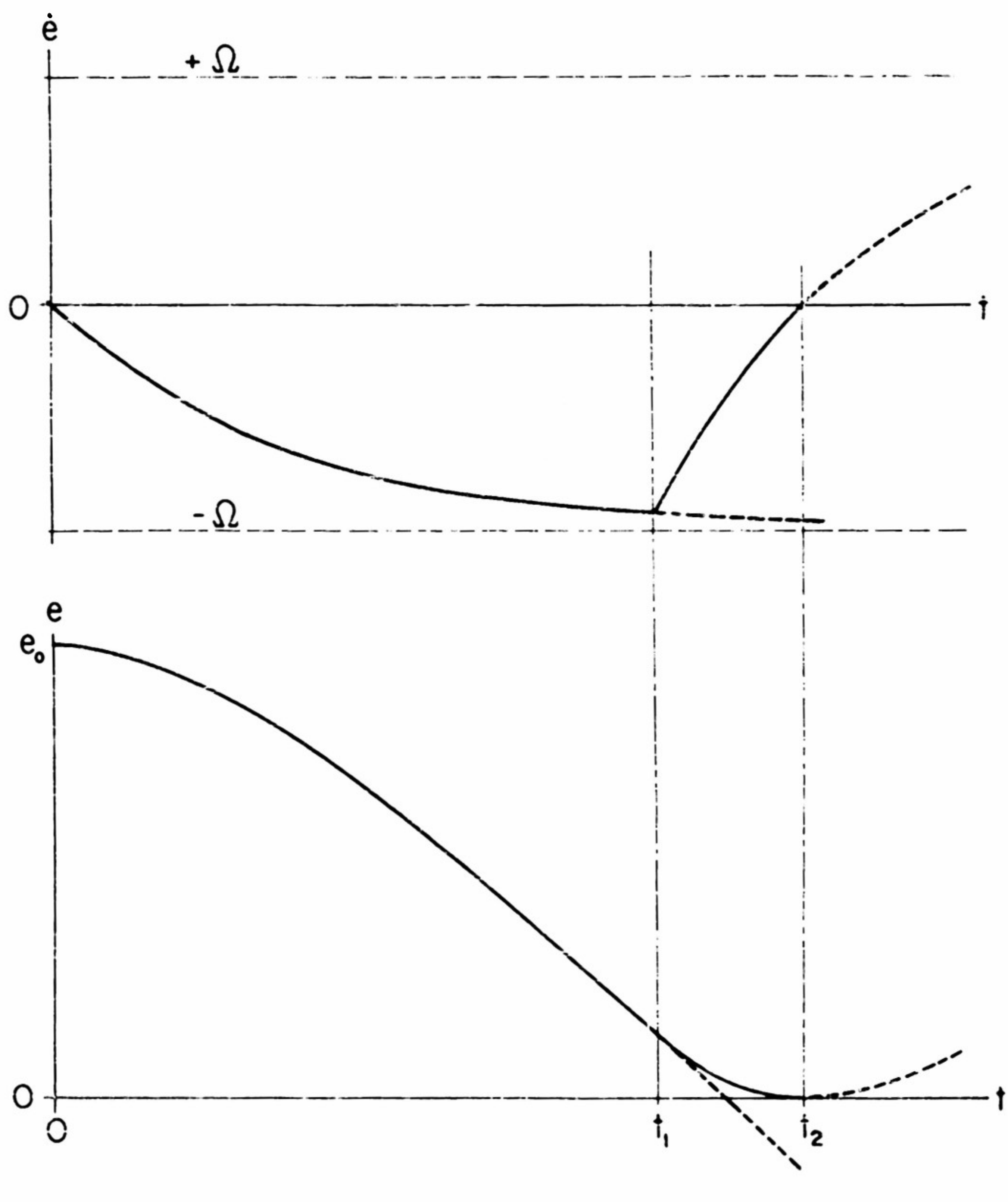


FIGURE 6A

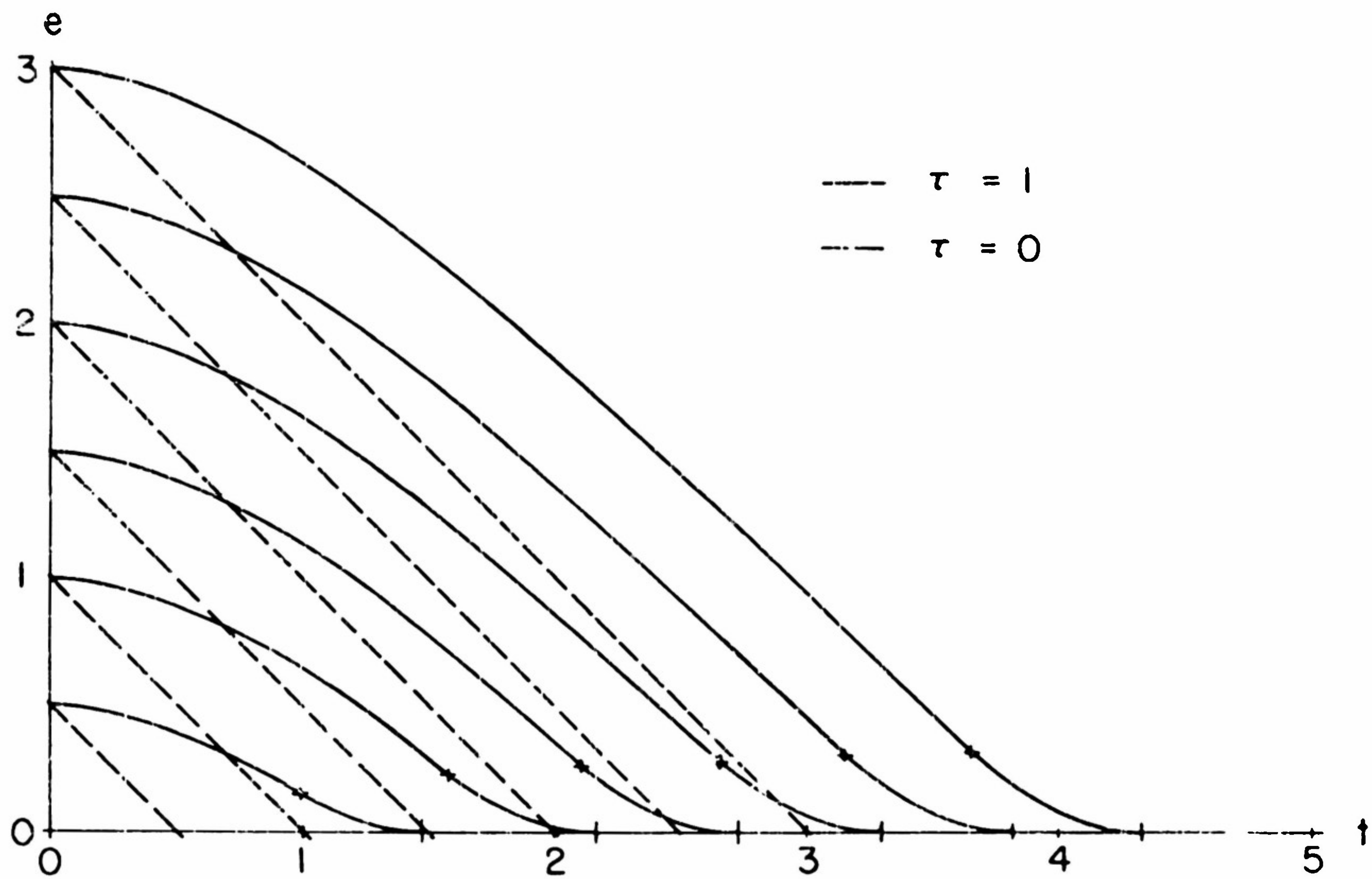


FIGURE 6B

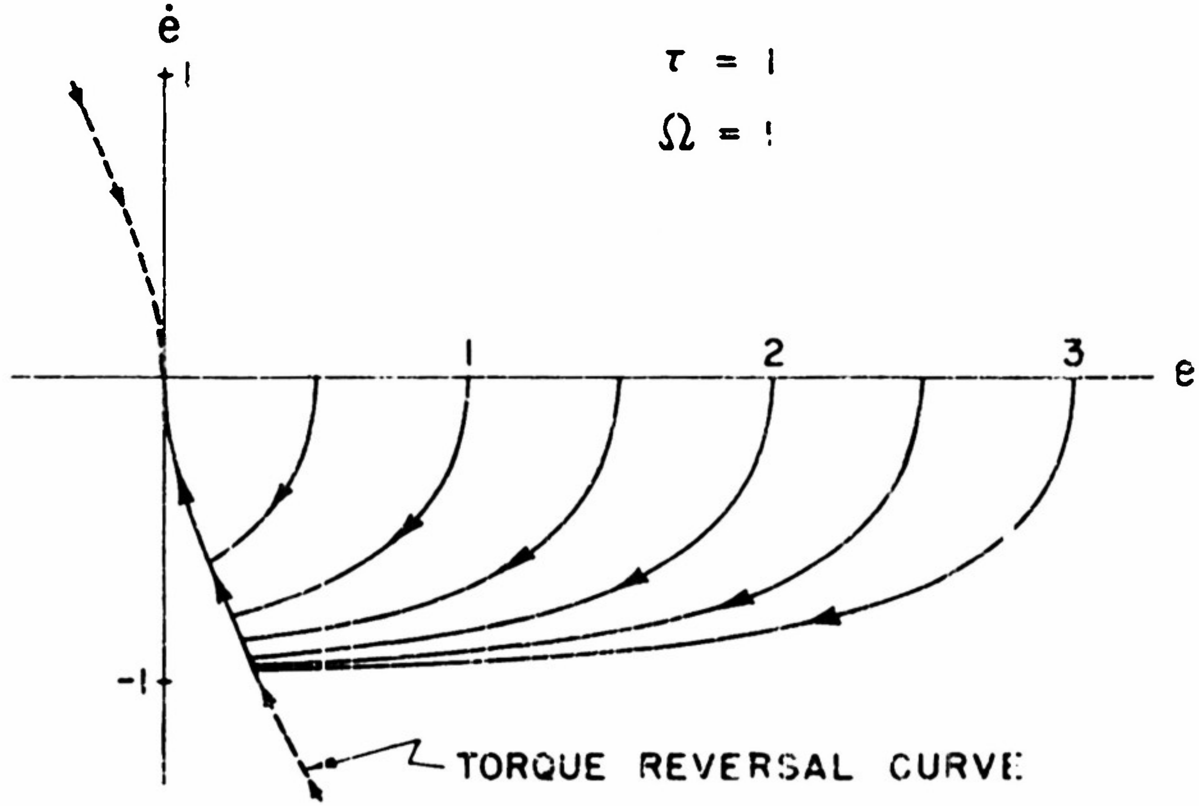
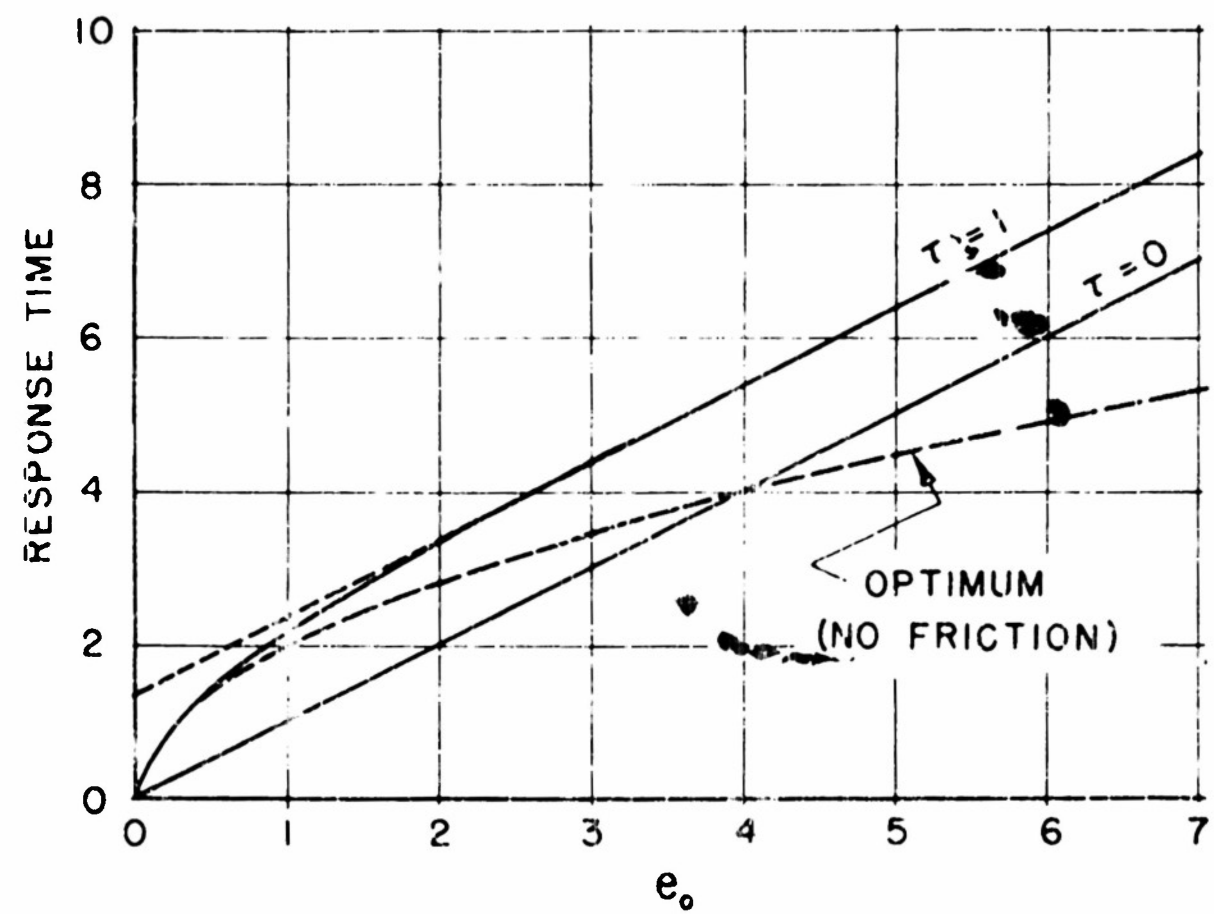


FIGURE 6C



10
8
6
4
2
0
RESPONSE TIME
0
1
2
3
4
5
6
7
e_o
$\tau_s = 1$
$\tau = 0$
OPTIMUM
(NO FRICTION)

FIGURE 6D

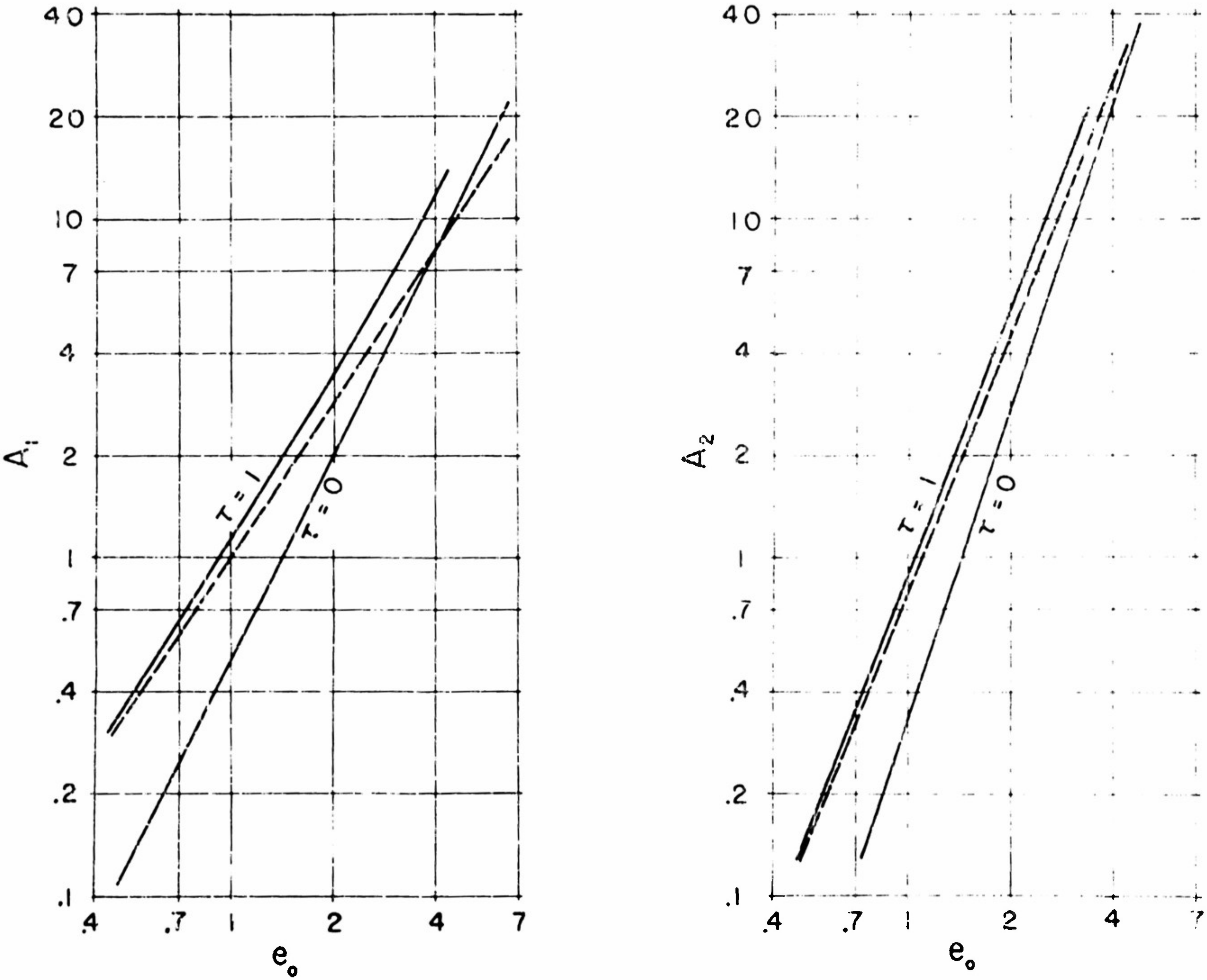
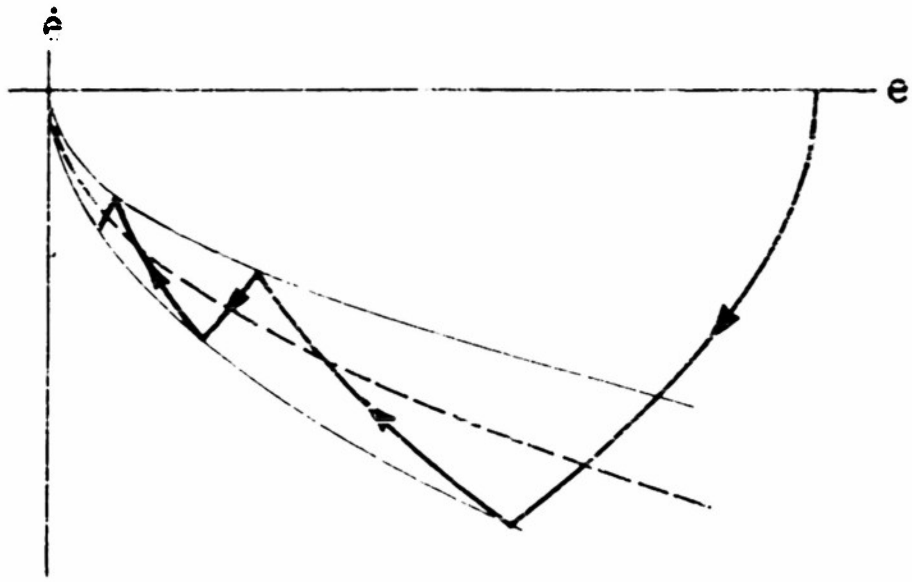
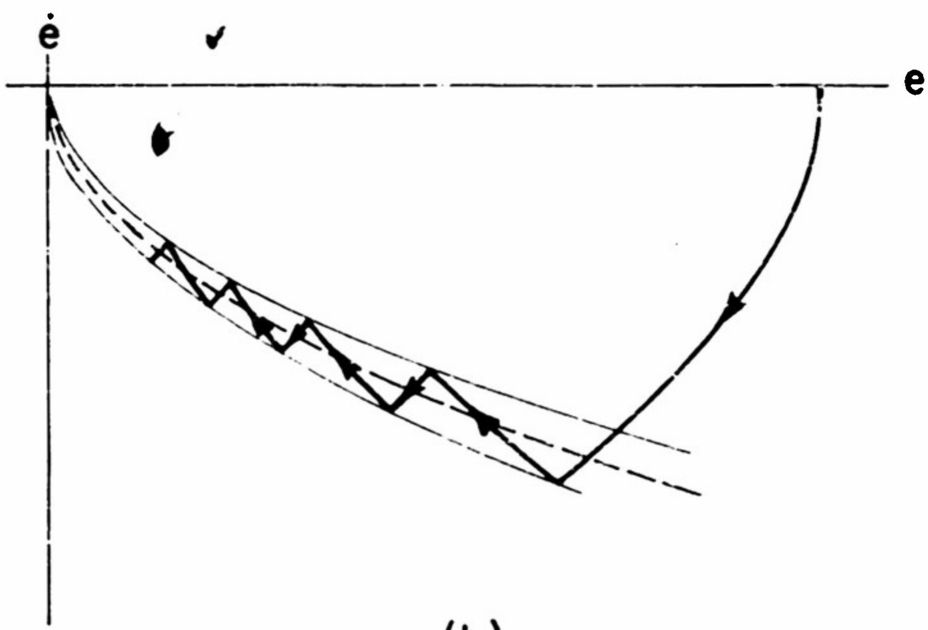


FIGURE 6E



(a)



(b)

FIGURE 7

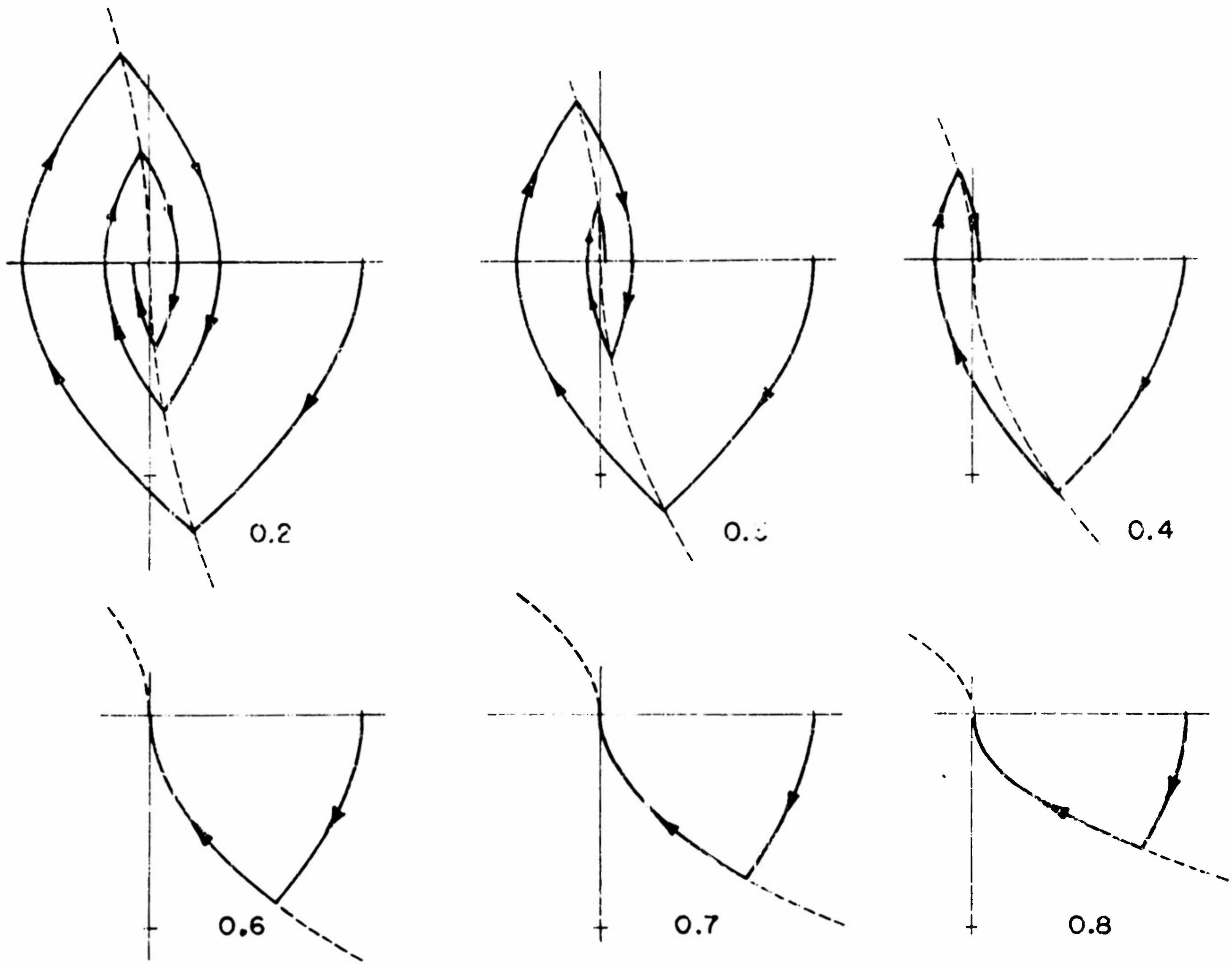


FIGURE 8

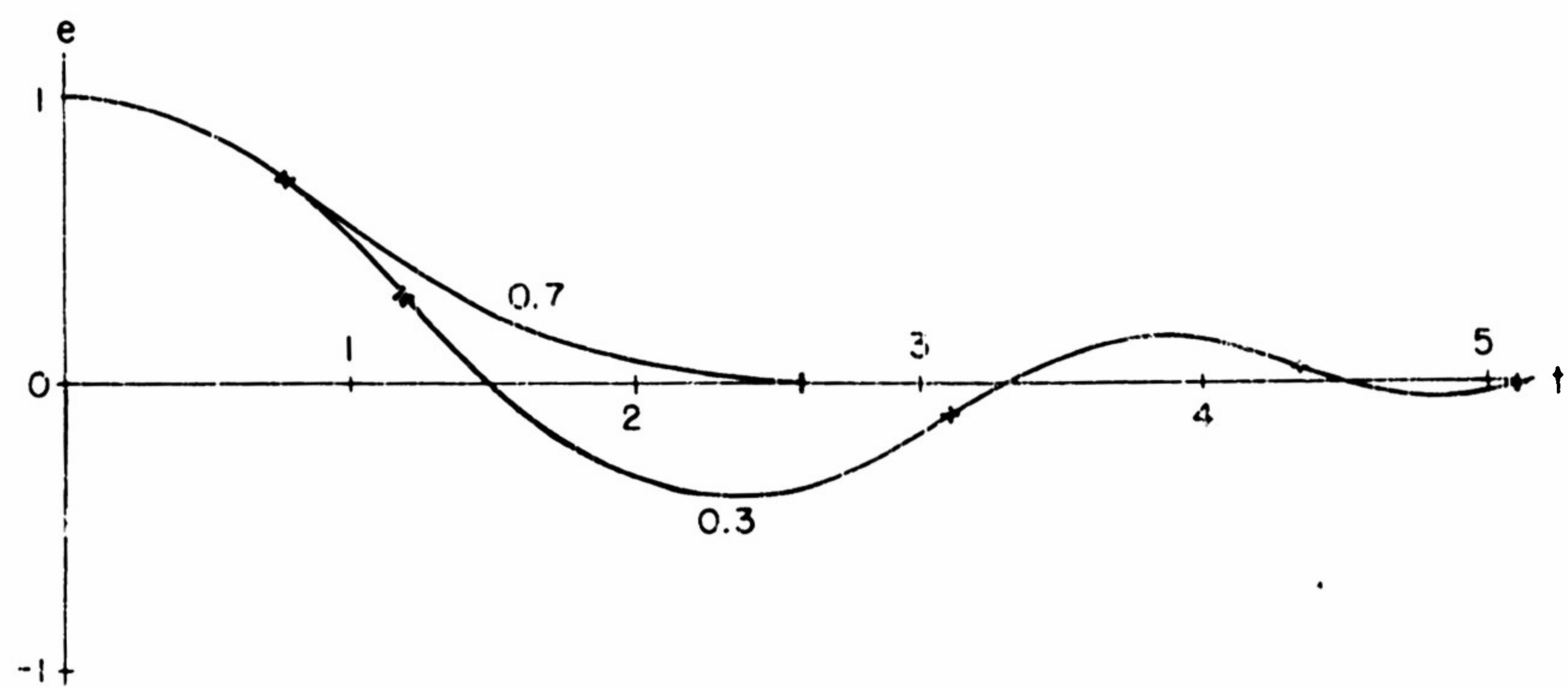
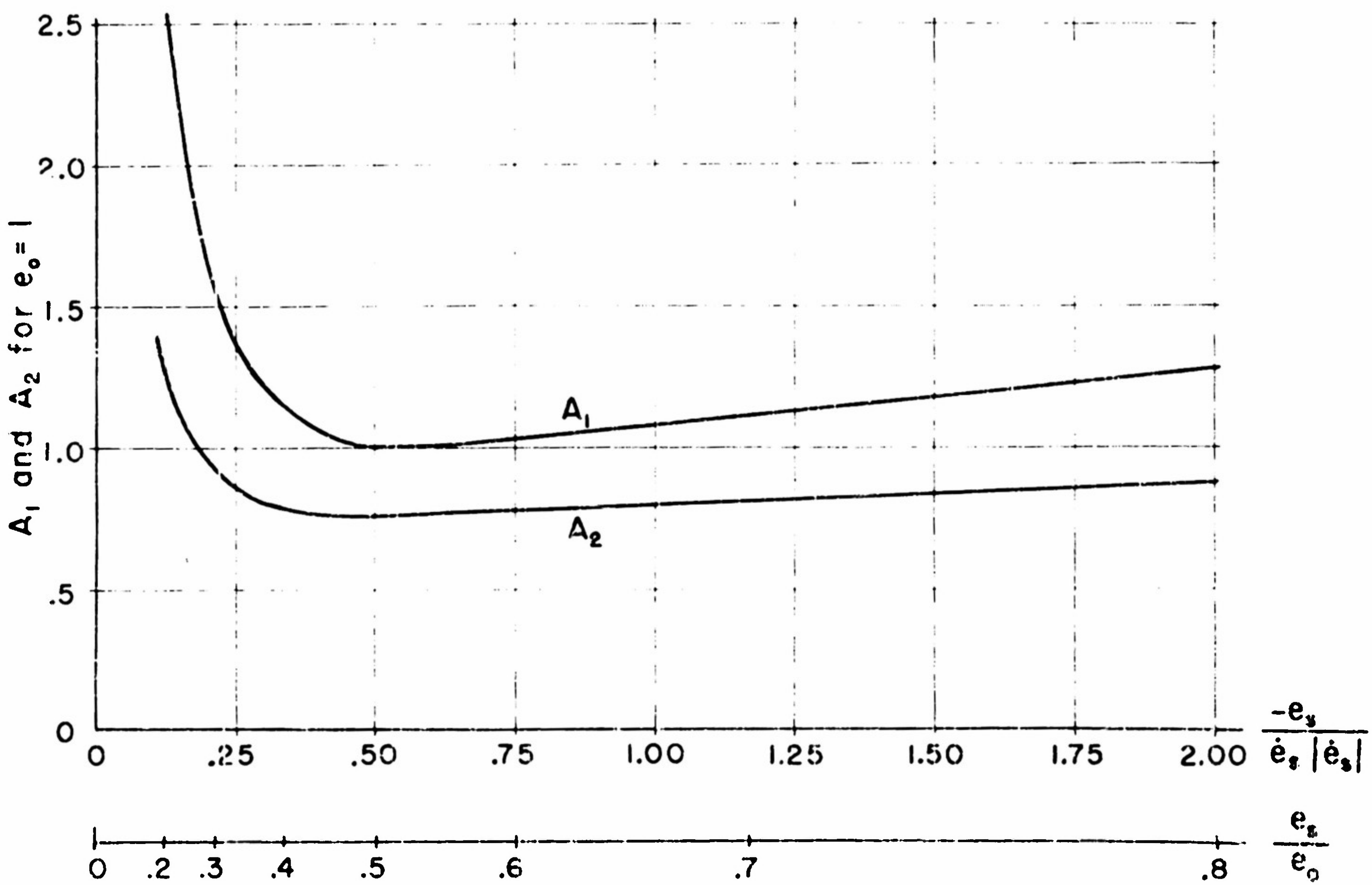
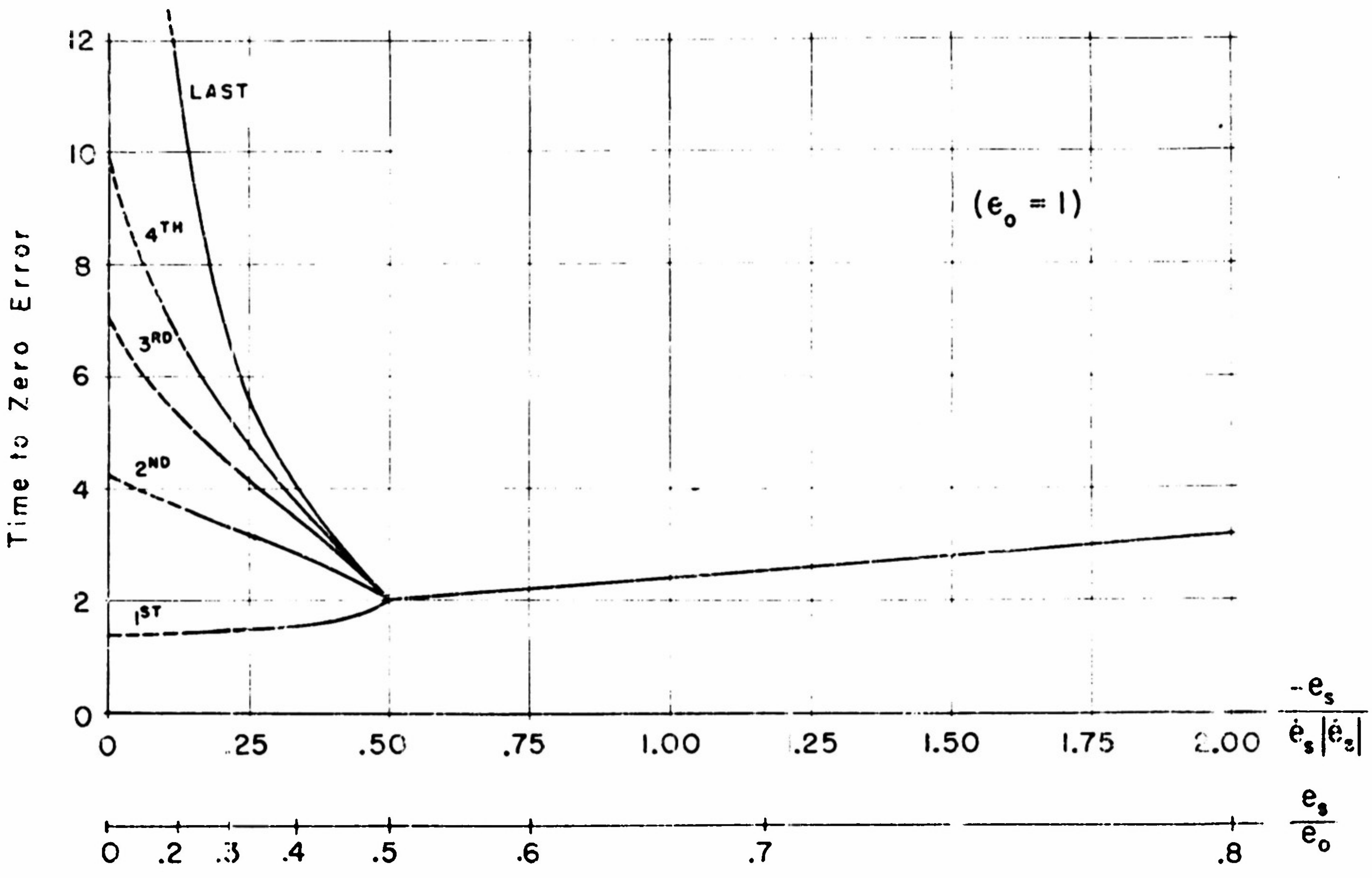


FIGURE 9



2.5
2.0
1.5
1.0
.5
0
A_1 and A_2 for $e_o = 1$
A_1
A_2
0
.25
.50
.75
1.00
1.25
1.50
1.75
2.00
$\frac{-e_s}{\dot{e}_s |\dot{e}_s|}$
0
.2
.3
.4
.5
.6
.7
.8
$\frac{e_s}{e_o}$

FIGURE 10



Time to Zero Error
LAST
4TH
3RD
2ND
1ST
$(\epsilon_o = 1)$
0
2
4
6
8
10
12
0
.25
.50
.75
1.00
.25
1.50
1.75
2.00
$\frac{-e_s}{\dot{e}_s|\dot{e}_z|}$
0
.2
.3
.4
.5
.6
.7
.8
$\frac{e_s}{e_o}$

FIGURE 11

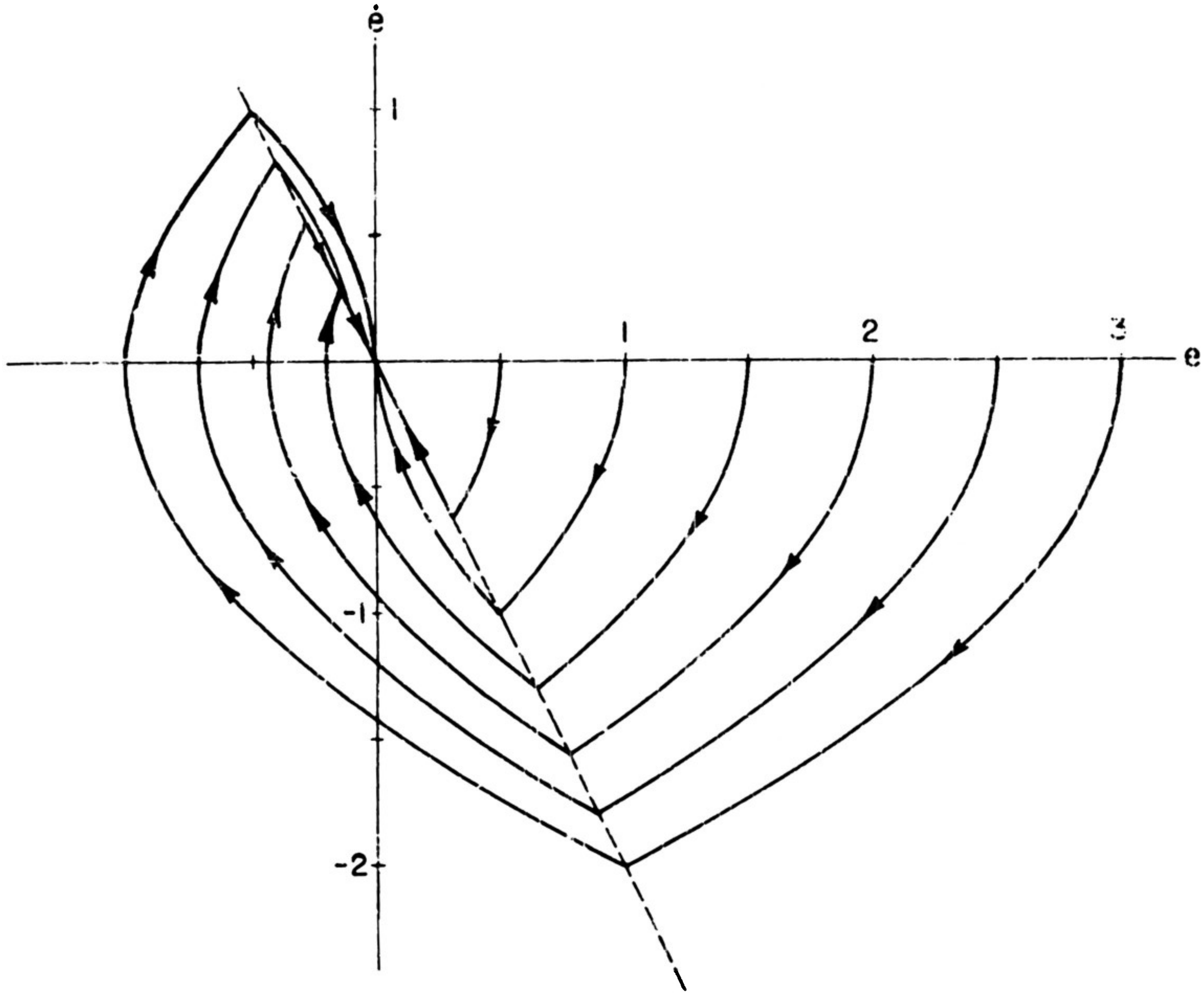


FIGURE 12

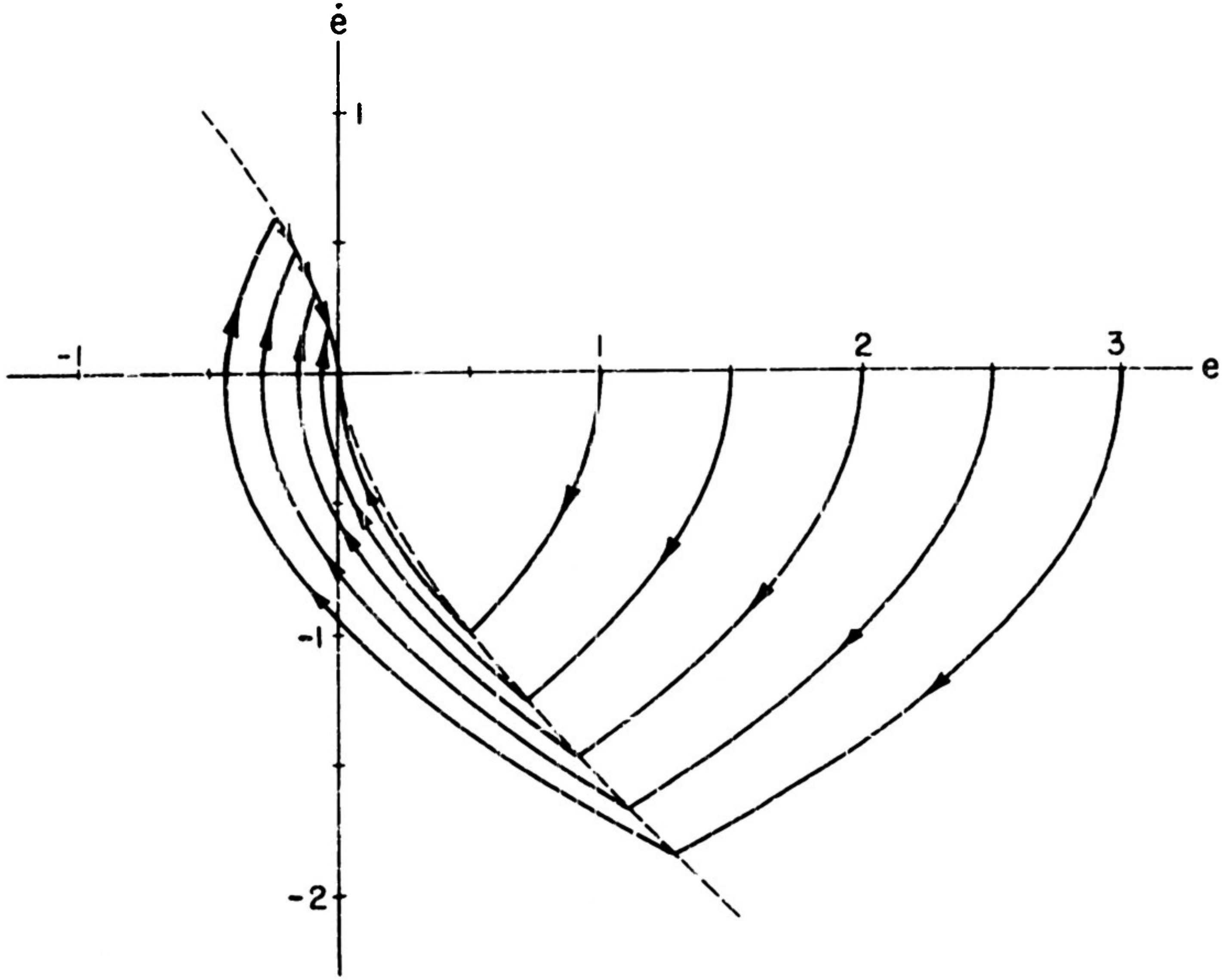


FIGURE 13

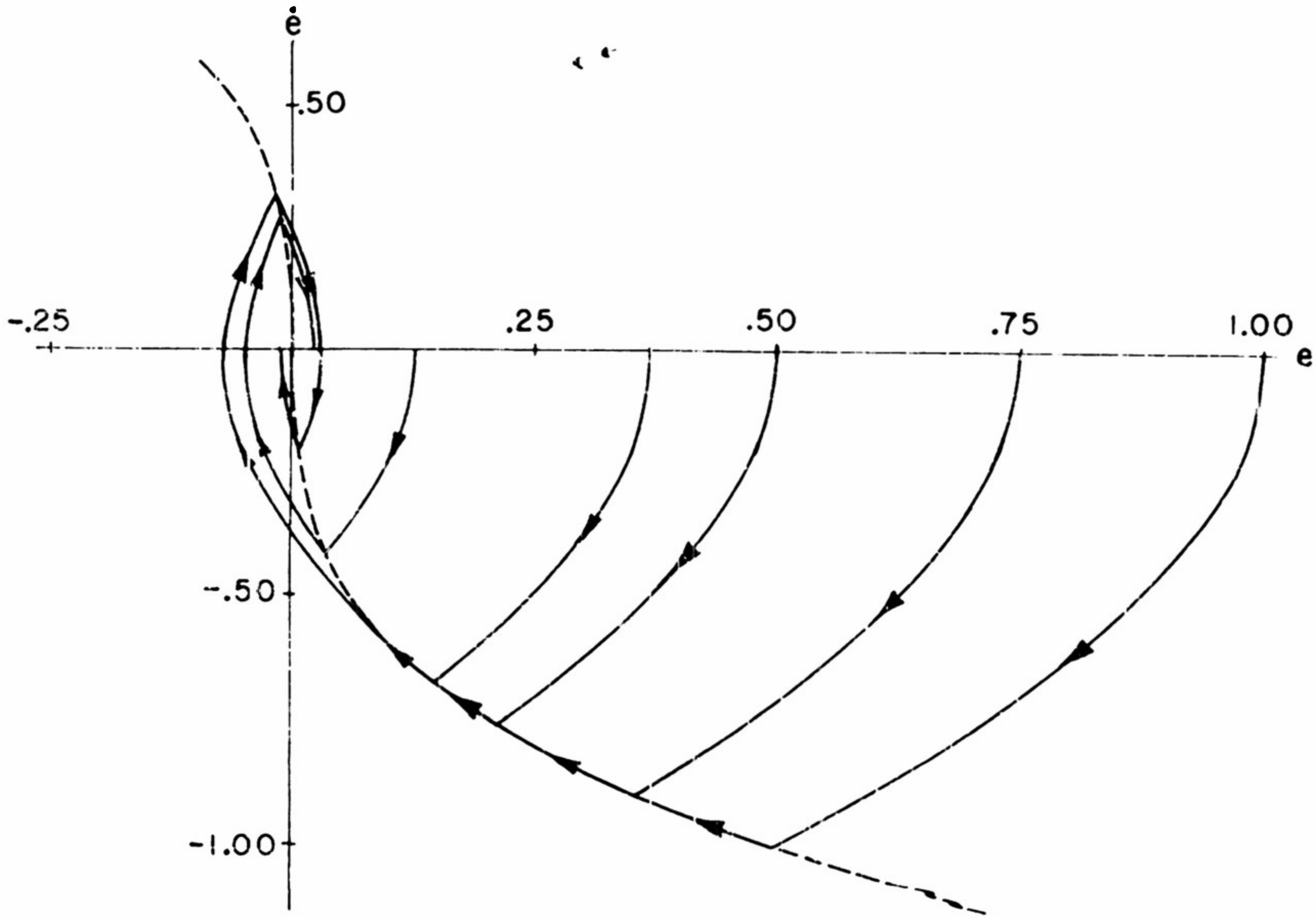


FIGURE 14

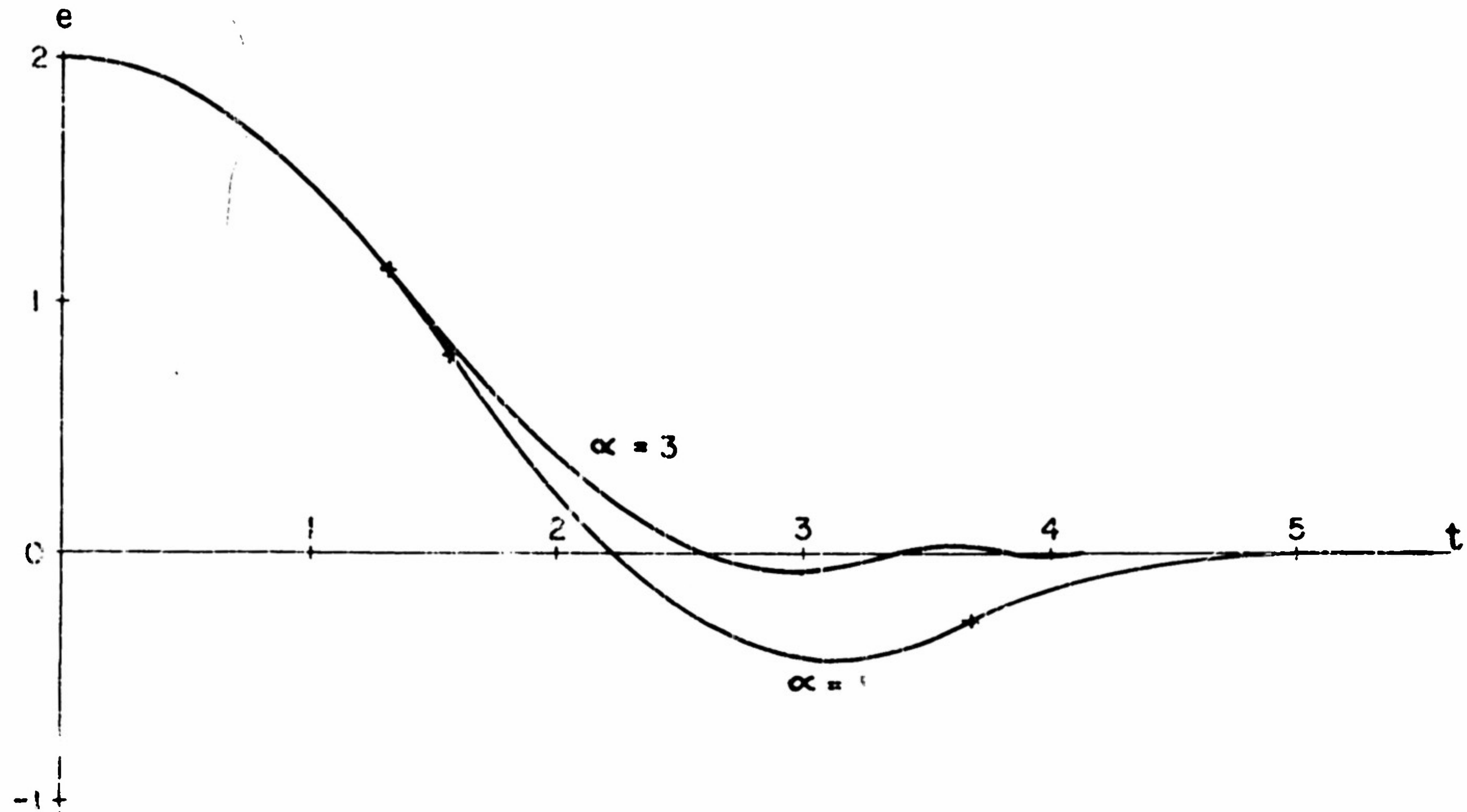


FIGURE 15

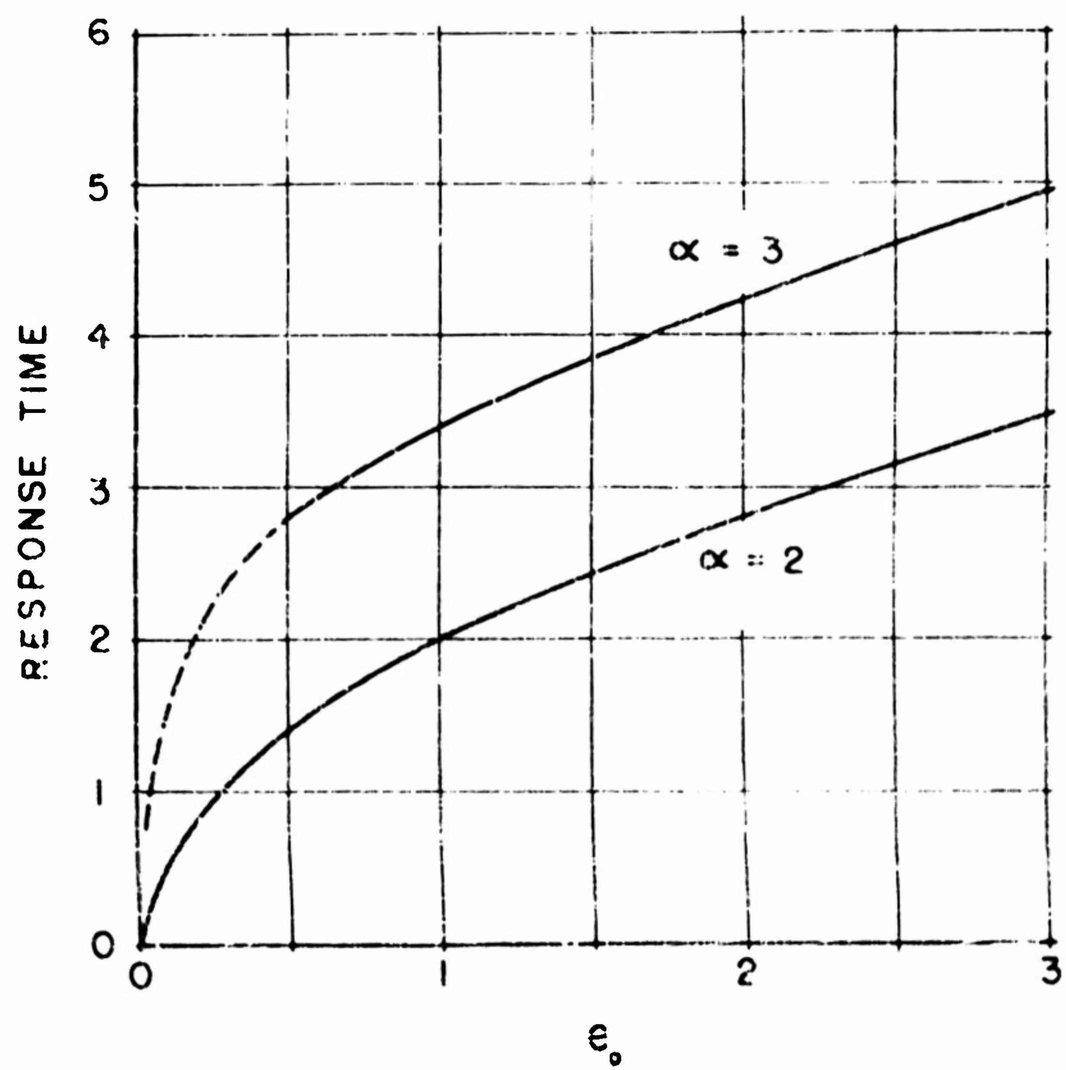


FIGURE 15A

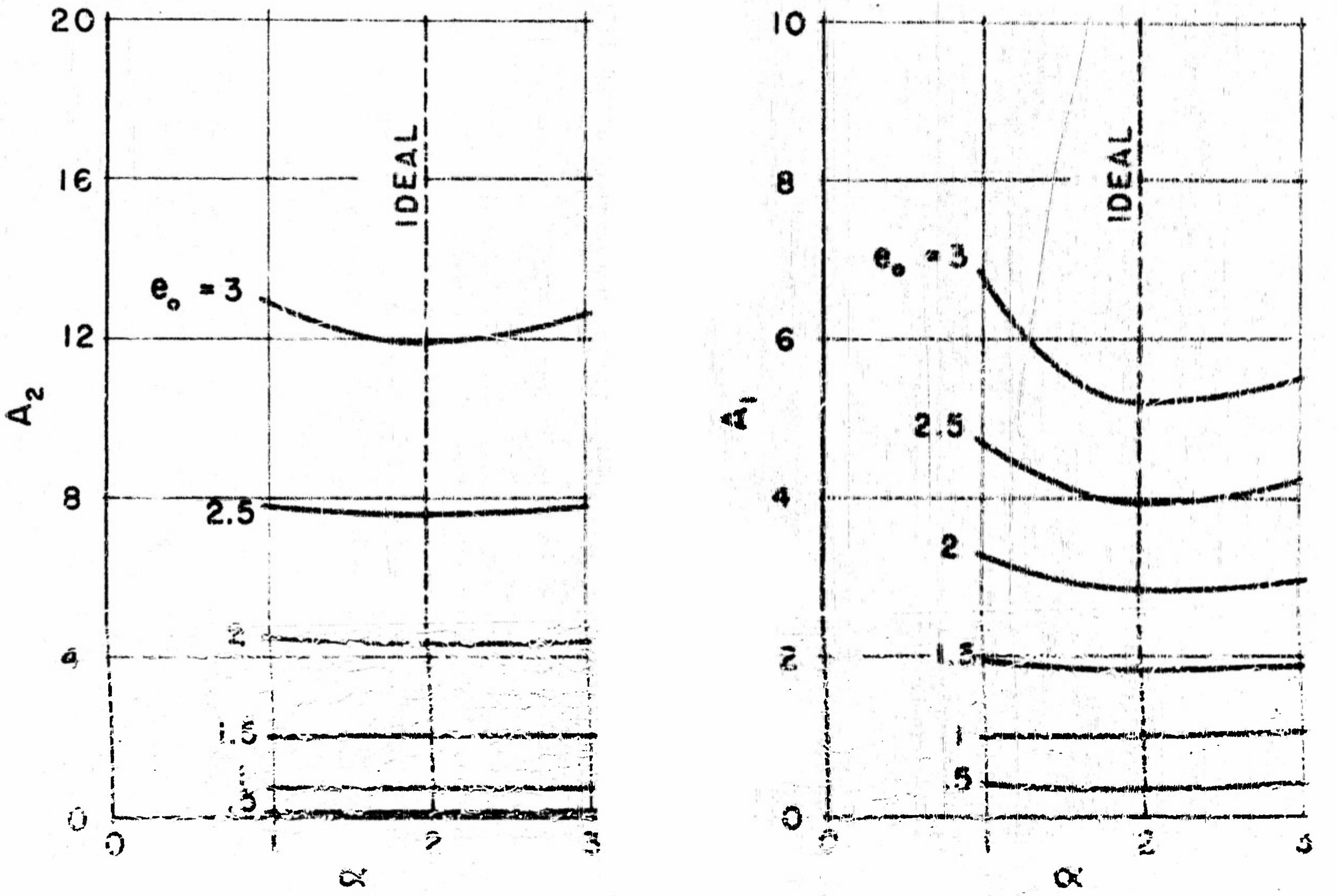

20
16
12
8
4
0
A_2
$e_o = 3$
2.5
2
1.5
IDEAL
0
1
2
3
α
10
8
6
4
2
0
A_1
$e_o = 3$
2.5
2
1.5
1
.5
IDEAL
0
1
2
3
α

FIGURE 16